THE HILBERT ε-OPERATOR
AND ITS SIGNIFICANCE FOR METAMATHEMATICS

by

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The Hilbert ε-Operator And Its Significance For Metamathematics

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In the first part of this thesis the ε-extension of first-order logic is reviewed and the classical consistency and conservancy results are proved. The Lattice of Chapters of ε-extended theories is defined and certain results are proved, including that the Lattice of Chapters of ε-extended theories is a proper subset of the ordinary Lattice of Chapters. Starting in Chapter 3 a new predicate logic is introduced, called Hologic, that uses a game-theoretic semantical scheme based on the ε-operator.

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CHAPTER

Chapter 1 The ε-Extension of First-Order Logic

The ε-extension of first order logic was introduced by David Hilbert in 1923 in [3]. It was motivated by his efforts to prove the consistency of mathematical analysis. Specifically, Hilbert introduced what he called a “logical operator" (denoted by ε) that replaces variables with “indeterminate constants" having the property that they may be interpreted as witnesses (or falsifiers) of the formula in which they occur.

Section 1.1 The Hilbert Extension

Let $\mathcal{L}$ denote the usual first-order logic with equality. In modern language Hilbert's ε-extension can be defined as follows. If $\varphi(\overline{x}, \overline{y})$ is a formula in $\mathcal{L}$ without quantifiers and $\overline{x}$, $\overline{y}$ are disjoint strings of variables comprising all the free variables of φ, then $\varepsilon\overline{y}\varphi$ denotes a string of $|\overline{y}|$ new function symbols, each with $|\overline{x}|$ argument places, where $|\overline{x}|$ and $|\overline{y}|$ denote the length of $\overline{x}$ and $\overline{y}$. Thus ε itself is an operator which operates on strings of variables and formulas and creates strings of function symbols.

> Definition 1.1.1 (The ε-Extension of First-order Logic)
>
> We denote by $\mathcal{L}^O$ (for “open logic”) first-order logic without quantifiers, and by $\mathcal{L}^\varepsilon$ the logic generated by the formulas of $\mathcal{L}^O$ and the operator ε (applied iteratively) and adding to its rules of proof the Hilbert axiom schema (i.e., rule)

$$\varphi(\overline{x}, \overline{y}) \to \varphi(\overline{x}, \varepsilon\overline{y}\varphi(\overline{x})). \tag{H}$$

> The deducibility relation in $\mathcal{L}^\varepsilon$ is denoted by $\vdash_\varepsilon$.

For readability, we may write $f(\overline{x})$ or $f_\varphi(\overline{x})$ for the string of terms $\varepsilon\overline{y}\varphi(\overline{x})$. Of course the operator ε allows us to make function symbols using formulas that already contain ε-terms. We may now treat quantifiers as informal abbreviations of the appropriate formulas with ε-terms. This is justified by Hilbert's observation that the schema (H) is equivalent in $\mathcal{L}$ to each of the following schemata:

$$\exists\overline{y}\varphi(\overline{x},\overline{y}) \leftrightarrow \varphi(\overline{x},\varepsilon\overline{y}\varphi(\overline{x})) \tag{1}$$

$$\forall\overline{y}\varphi(\overline{x},\overline{y}) \leftrightarrow \varphi(\overline{x},\varepsilon\overline{y}\neg\varphi(\overline{x})). \tag{2}$$

These equivalent forms of (H) allow us to interpret quantifiers as abbreviations of certain quantifier-free formulas, as follows: We interpret the symbols of the sequence $\varepsilon\overline{y}\varphi$ to be witnesses for the existential quantifier $\exists\overline{y}$ in the formula $\exists\overline{y}\varphi(\overline{x},\overline{y})$, if any exist, and otherwise picking arbitrary objects. Similarly, we understand the symbols of the sequence $\varepsilon\overline{y}\neg\varphi$ to be Skolem functions picking falsifiers for the universal quantifier $\forall\overline{y}$ in the formula $\forall\overline{y}\varphi(\overline{x},\overline{y})$, if any exist, and otherwise picking arbitrary objects.

Our L^{ε} is closed under the ε-operator, but there are no quantifiers in it. However, every formula of $\mathcal{L}$ can be translated uniquely into a formula of $\mathcal{L}^{\varepsilon}$ (applying (1) and (2) and operating from inside out), and this is the reason we say that $\mathcal{L}^{\varepsilon}$ is an extension of $\mathcal{L}$.

> Remark 1.1.2
>
> We can say that the ε-operator is a formalization of the idea of Skolem functions. We refer to the process of eliminating quantifiers in a formula of $\mathcal{L}^{\varepsilon}$ by means of ε-terms as *epsilonizing*. How one chooses to epsilonize a given formula, and in particular the order in which variables are replaced with ε-functions, may affect what the functions themselves are as well as their arity. Hence, epsilonizing in different ways may result in a variety of distinct but logically equivalent formulas.

Now we will show that the consistency of the ε-extension is an easy consequence of the consistency of Henkin's rule. For simplicity we consider only logic without equality, that is, the axioms of equality are treated as mathematical axioms that can appear in theories, but not as axioms of logic.

Theorem 1.1.3 (Consistency of Henkin's Rule)

If T is a consistent theory, φ is a formula of T with free variables $\overline{x}$, $\overline{c}_\varphi$ are constant symbols that do not appear in T, and $|\overline{c}_\varphi| = |\overline{x}|$, then $T \cup \{\varphi(\overline{x}) \to \varphi(\overline{c}_\varphi)\}$ is also consistent.

Proof: Suppose otherwise. Then

$$T \vdash \exists\overline{x}\varphi(\overline{x}) \wedge \neg\varphi(\overline{c}_\varphi). \tag{3}$$

Since $\overline{c}_\varphi$ does not appear in T we can replace the constants of $\overline{c}_\varphi$ in the proof of (3) by variables $\overline{y}$ that do not appear in that proof. So we get $T \vdash \exists\overline{x}\varphi(\overline{x}) \wedge \neg\varphi(\overline{y})$ and $T \vdash \exists\overline{x}\varphi(\overline{x}) \wedge \forall\overline{y}\neg\varphi(\overline{y})$, so T is inconsistent, contrary to the assumption.

The formula $\varphi(\overline{x}) \to \varphi(\overline{c}_\varphi)$ is called Henkin's rule, and the terms $\overline{c}_\varphi$ are called Henkin's constants.

Theorem 1.1.4 (Consistency of the ε-extension)

If T is a consistent theory in $\mathcal{L}$ and if $\varphi(\overline{x}, \overline{y})$ is a formula in the language of T, then

$$T \cup \{\varphi(\overline{x}, \overline{y}) \to \varphi(\overline{x}, \varepsilon\overline{y}\varphi(\overline{x}))\} \tag{4}$$

is also consistent.

Proof: By Theorem 1.1.3 we can assume T is closed under Henkin's Rule. Let M be a model of the Henkin extension of T whose elements are the Henkin constants. For any $\varphi(\overline{x}, \overline{y})$ and any $|\overline{x}|$-tuple $\overline{c}$ of Henkin constants, let f be a function such that $f(\overline{c}) := \overline{c}_{\varphi(\overline{c},\overline{y})}$. Then the theory (4) is true in the model (M, f) if $\varepsilon\overline{y}\varphi$ is interpreted as f.

Definition 1.1.5 (Theories and ε-Theories)

If T is a theory in $\mathcal{L}$, we will identify T with the deductive closure of a set A of axioms of T in $\mathcal{L}$, i.e., T is the smallest set of formulas such that T contains A, and if $A \vdash \varphi$ then $\varphi \in T$. We denote by T^{ε} the deductive closure of A in $\mathcal{L}^{\varepsilon}$, i.e., T^{ε} is the smallest set of formulas such that T^{ε} contains the set A of axioms of T, and if $A \vdash_{\varepsilon} \varphi$ then $\varphi \in T^{\varepsilon}$.

Of course we have $T \subseteq T^{\varepsilon}$, but in general $T \neq T^{\varepsilon}$. However $T^{\varepsilon} = (T^{\varepsilon})^{\varepsilon}$.

Hilbert proved in the 1920's that $\mathcal{L}^{\varepsilon}$ is a conservative extension of $\mathcal{L}$, in the sense of the following theorem.

Theorem 1.1.6 (Hilbert's Second ε-Theorem)

If φ is a formula in the language of T and $T \vdash_{\varepsilon} \varphi$, then $T \vdash \varphi$.

Proof: We can assume without loss of generality that φ is a sentence (i.e., has no free variables). Suppose $T \vdash_{\varepsilon} \varphi$ but $T \nvdash \varphi$. Then $S = T \cup \{\neg\varphi\}$ is consistent. But $S^{\varepsilon} \supseteq T^{\varepsilon}$, and hence $S \vdash_{\varepsilon} \varphi \wedge \neg\varphi$, contrary to the consistency of the ε-extension.

Section 1.2 The Lattice **LC** of Chapters of Mathematics.

In this section we recall some definitions from the paper of Mycielski, Pudlak, and Stern [6]. These definitions pertain to the usual first-order logic $\mathcal{L}$. In the next section we will discuss similar concepts for the logic $\mathcal{L}^{\varepsilon}$.

An n-dimensional interpretation I of a language L_1 into a language L_2 is an assignment of the relations, functions, and of the equality symbol of L_1 to some formulas in L_2, and of the argument places of those symbols to some disjoint n-tuples of free variables of the assigned formulas. In the case of a function symbol f the formula assigned to f corresponds to the equation $f(\overline{x}) = y$, and so it must have an additional n-tuple of free variables (for the variable y). We call I an interpretation *with parameters* if the assigned formulas of L_2 have additional free variables (called parameters) not in the n-tuples corresponding to the original variables. If σ is a sentence in L_1, then σ^I denotes the existential closure of the formula obtained by translating σ into the language L_2 by means of I in the natural way. Since σ was a sentence, σ^I is also a sentence. We say that σ is interpretable in T_2 if and only if there exists an I for the language of σ such that σ^I is a theorem of T_2.

For theories T_1 and T_2, we will write $T_1 \preceq T_2$ and say that T_1 is *locally interpretable* in T_2 if every theorem of T_1 is interpretable in T_2. (Note that I and n can depend on the theorem.) We will say that T_1 and T_2 are *equi-interpretable*, denoted $T_1 \simeq T_2$, if $T_1 \preceq T_2$ and $T_2 \preceq T_1$.

We observe that $\simeq$ is an equivalence relation on theories. From now on, as in [6], we assume that all theories are consistent and do not have 1-element models. The equivalence classes are called *chapters*. The equivalence class of a theory T will be denoted by $|T|$. (However each $|T|$ is a proper class in the sense of set theory, we will see in a moment that there are at most continuum many such classes. It was shown in [6] that there are exactly $2^{\aleph_0}$ chapters.) The relation $\preceq$ induces a partial order of the chapters. The class of all these equivalence classes will be denoted by **LC**.

Two sentences are said to be of the same shape if each can be obtained from the other by changing only its function symbols and relation symbols. It is clear that there are only countably many shapes of sentences, and the interpretability of a sentence σ

in a theory T depends only on the shape of σ. Thus any chapter $|T|$ can be represented by the set of shapes of sentences interpretable in T. It follows that there are only continuum many chapters.

Notice that $\preceq$ induces a partial order $\leq$ in **LC**, that the set of shapes of all consistent sentences constitutes a maximal element of **LC**, and that for any set A of chapters the intersection of the set of sets of shapes representing the chapters of A represents a chapter, and this chapter is the largest lower bound of A in **LC**.

To summarize, we have:

1. $\preceq$ is reflexive and transitive, and hence $\leq$ is a partial ordering.
2. $S \simeq T$ if and only if $|S| = |T|$.

Notice also that the least chapter containing a given shape of a consistent sentence is a compact element of **LC**. Hence every chapter is a join of compact chapters. This implies that **LC**, *ordered by* $\leq$, *is an algebraic lattice*. Mycielski has shown also that **LC** *is a distributive lattice* (see [6]). It follows that **LC** *is also a Brouwerian lattice*.

Section 1.3 The Lattice of ε-Chapters

Now we will define similar concepts for the logic $\mathcal{L}^{\varepsilon}$. Notice that every $\mathcal{L}^{\varepsilon}$-theory is also an $\mathcal{L}$-theory. And if T is a finitely axiomatizable theory then T^{ε} is also finitely axiomatizable in the logic $\mathcal{L}^{\varepsilon}$ (by the same axioms as T).

We claim (informally) that the usual sense in which mathematicians (who are not logicians) use the phrase "S is a mathematical theory" is reasonably defined by "S is a theory of the form T^{ε}." This is because in mathematical practice everybody accepts the Hilbert rule (H) in the sense that they accept a principle of logical choice.

Let $\mathbf{LC}^{\varepsilon}$ be the class of chapters of the form $|T^{\varepsilon}|$. By the above remarks it seems that a study of $\mathbf{LC}^{\varepsilon}$ will be motivated even better than the study of **LC**.

The theory of the lattice **LC** was developed in [6] and subsequent papers. I will not be able to do as much for $\mathbf{LC}^{\varepsilon}$. In fact I will limit my work to the following basic remarks and theorems.

Theorem 1.3.1

The relation $\leq$ restricted to $\mathbf{LC}^{\varepsilon}$ is also a partial ordering of a complete, distributive algebraic lattice.

> Proof: If A is a set of chapters, for every $a \in A$ we can choose a theory T_a such that $|T_a| = a$ and the languages of the theories T_a are disjoint. Let T be the ε-theory based on a set of axioms $\bigcup_{a\in A} T_a$. Thus $|T|$ belongs to $\mathbf{LC}^{\varepsilon}$ and it is easy to check that $|T|$ is the least upper bound of A. Hence $\mathbf{LC}^{\varepsilon}$ is a complete lattice.
>
> It is also clear that for every theory T the chapter $|T^{\varepsilon}|$ is a join of chapters of the form $|S^{\varepsilon}|$, such that each S^{ε} is axiomatizable in $\mathcal{L}^{\varepsilon}$ by one sentence. It is easy to check that $|S^{\varepsilon}|$ are compact elements of the lattice. Thus $\mathbf{LC}^{\varepsilon}$ is an algebraic lattice. Lattice distributivity can be derived from distributivity in propositional calculus in the same way as in [6].

Dale Myers has shown that $\mathbf{LC}^{\varepsilon}$ is a proper subset of **LC**:

Theorem 1.3.2

The chapter of endless linear orders is not in $\mathbf{LC}^{\varepsilon}$.

> *Outline of proof*: Let α be the conjunction of the axioms of the theory T of endless linear orders, and let β be the sentence
>
> $$\alpha \wedge \forall x (x < f(x)).$$

Of course β is interpretable in T^ε. Any model of β has infinitely many types of pairs of elements. Indeed, the binary formulas

$$f(x) = y, f(f(x)) = y, f(f(f(x))) = y, \ldots$$

are mutually exclusive in the sense that no pair (x, y) can satisfy two of them since $f^k(x) < f^n(x)$ for $k < n$. On the other hand, some models of T have only finitely many types of n-tuples for every n. For example, $\langle \mathbb{Q}, < \rangle$ has this property. Hence every sentence which is interpretable in T has models which have only finitely many binary types. Indeed, if the interpretation is n-dimensional, then it has no more binary types than the number of types of sequences of length $2n$. Thus β is not interpretable in T.

Theorem 1.3.3

For any two theories U and V, if $U \preceq V$ then $U^\varepsilon \preceq V^\varepsilon$.

Proof:

Let σ be a sentence in the language of U^ε of the form

$$\alpha \wedge \bigwedge_{i=1}^{n} \forall \overline{x}_i \forall \overline{y}_i [\phi_i(\overline{x}_i, \overline{y}_i) \rightarrow \phi_i(\overline{x}_i, \varepsilon \overline{y}_i \phi_i(\overline{x_i}))],$$

where α is any sentence that is a theorem of U. Of course σ is a theorem of U^ε. Moreover, for every theorem τ of U^ε there exists a σ as above such that $\sigma \rightarrow \tau$ is a theorem of logic. Thus is suffices to prove that there exists an interpretation I such that σ^I is a theorem of V^ε. By the assumption of the theorem we have an interpretation I of the language of σ such that α^I is a theorem of V. Let $\overline{p}$ be the sequence of parameters of this interpretation. Then σ^I has a prefix $\exists\overline{p}$. Omitting this turns σ^I into a formula $\psi(\overline{p})$ whose free variables are exactly those of $\overline{p}$. We form the sentence $\psi(\varepsilon\overline{p}\psi)$ and it is clear that this is an interpretation of σ. Hence it remains to show that it is also a theorem of V^ε. Notice that $\psi(\varepsilon\overline{p}\psi)$ is a conjunction of two sentences (similar to σ). The first is a theorem of V^ε since α^I is a theorem of V. The second is a conjunction of universal closures of instances of (H). Hence it is also a theorem of V^ε.

Corollary 1.3.4

$|T| \in \mathbf{LC}^\varepsilon$ if an only if $T^\varepsilon \preceq T$.

Proof: If $|T| \in \mathbf{LC}^\varepsilon$ then $|T| = |S^\varepsilon|$ for some theory S. Hence $T \preceq S^\varepsilon$. By the previous theorem, $T^\varepsilon \preceq S^\varepsilon$. Hence $|T^\varepsilon| \leq |S^\varepsilon| = |T|$ and $T^\varepsilon \preceq T$.

Vice versa, if $T^\varepsilon \preceq T$ then $|T| = |T^\varepsilon| \in \mathbf{LC}^\varepsilon$.

We observe that the converse of Theorem 1.3.3 fails. For example, if V is the theory of endless linear orders and $U = V^\varepsilon$, then

$$U^\varepsilon \preceq V^\varepsilon$$

holds, but by the theorem of Myers $U \npreceq V$.

Definition 1.3.5

A first-order theory T is said to have *definable selectors* if for any formula $\varphi(\overline{x}, \overline{y})$ there is a formula $\psi(\overline{x}, \overline{y})$ such that

$$T \vdash \forall\overline{x}[\exists\overline{y}\varphi(\overline{x}, \overline{y}) \to \exists!\overline{y}(\varphi(\overline{x}, \overline{y}) \land \psi(\overline{x}, \overline{y}))]. \tag{5}$$

It is clear that every theory of the form T^{ε} is a theory with definable selectors. For example, as is well known, the following theories have definable selectors: RCF (the theory of real closed fields), PA (Peano Arithmetic), and ZF+V=OD (set theory with the claim that the universe is ordinal-definable).

Theorem 1.3.6

If T has definable selectors then $|T| = |T^{\varepsilon}|$.

Proof: Suppose T has definable selectors. Since $T^{\varepsilon} \vdash T$ it is sufficient to establish $T^{\varepsilon} \preceq T$. To prove this we construct, inductively, an interpretation J (1-dimensional and with one parameter) of the language of T^{ε} into the language of T, such that for every theorem σ of T^{ε}, σ^{J} is a theorem of T. Let us add to the language of T a new constant c. (This augmented theory is still a theory with selectors. Later we will get rid of c.) Now we will construct an auxiliary interpretation I, one-dimensional and without parameters, of T^{ε} into T with the constant c. The symbols of T remain unchanged. Now we will translate the equations $\varepsilon\overline{y}\chi(\overline{x}) = \overline{y}$ for any formula χ in the language of T^{ε} into the language of T plus c. We do this inductively. Suppose we have constructed enough of I to translate χ into a formula ϕ in the language of T plus c. (Since I has no parameters χ and ϕ have the same free variables.) Then we express $\varepsilon\overline{y}\chi(\overline{x}) = \overline{y}$ by means of the formula

$$(\phi(\overline{x}, \overline{y}) \rightarrow \psi(\overline{x}, \overline{y})) \wedge (\neg\phi(\overline{x}, \overline{y}) \rightarrow \overline{y} = \overline{c}),$$

where ψ is as in (5) for the language with c, and where $\overline{c}$ is a string of length $|\overline{y}|$ all of whose terms are the constant c. It

> is clear from this inductive construction of I that, for every σ belonging to T^{ε}, σ^{I} is a theorem of T with the constant c. It remains to get rid of c. Since T has no assumptions about c we can replace c in σ^{I} by a free variable, and quantify it existentially. In this way we get the interpretation J such that σ^{J} is a theorem of T.

Let S and T be first-order theories, and assume that the sets of constants, relation symbols (including equality), and function symbols of S and T are disjoint. We recall from [6] that we define $S + T$ to be the theory whose axioms are $\alpha \wedge \beta$, where $\alpha \in S$ and $\beta \in T$. Similarly, we define $S \cdot T$ to be the theory whose axioms are $\alpha \vee \beta$, where $\alpha \in S$ and $\beta \in T$.

We saw that **LC**$^{\varepsilon}$ is a lattice and we will prove in the next section that the meets in **LC**$^{\varepsilon}$ are the same as the meets in **LC**, that is

$$|T^{\varepsilon}| \wedge |S^{\varepsilon}| = |T^{\varepsilon}| \underset{\varepsilon}{\wedge} |S^{\varepsilon}|. \tag{6}$$

However, **LC**$^{\varepsilon}$ is a proper subset of **LC** and it seems that the joins in **LC**$^{\varepsilon}$ can be larger than the joins in **LC**. To be precise,

Problem 1: Do there exist theories S and T such that

$$|T^{\varepsilon}| \vee |S^{\varepsilon}| < |T^{\varepsilon}| \overset{\varepsilon}{\vee} |S^{\varepsilon}|?$$

Section 1.4 Chapters as Closed Sets of Shapes

There is a second way of defining **LC** given in [6] which we will extend in a natural way to a definition of **LC**$^{\varepsilon}$.

Definition 1.4.1

Given a formula φ of first-order logic, we define the *shape* of φ to be the equivalence class of formulas that can be obtained from φ by uniformly changing only the relation and function symbols, but preserving their arities and keeping distinct symbols distinct. Given a theory T, we denote by $Sh(T)$ the set of all shapes of sentences interpretable in T.

Definition 1.4.2

For any set X of shapes, choose a sentence α_x having shape x for each $x \in X$, such that the sentences α_x have no relation or function symbols in common. We will denote by $Th(X)$ the deductive closure in $\mathcal{L}$ of $\{\alpha_x : x \in X\}$, and by $Th^{\varepsilon}(X)$ the deductive closure in $\mathcal{L}^{\varepsilon}$ of $\{\alpha_x : x \in X\}$. We then define closure operators $\overline{X} = Sh(Th(X))$ and $\overline{X}^{\varepsilon} = Sh(Th^{\varepsilon}(X))$.

The closure operators defined above are of finite type, that is,

$$\overline{X} = \bigcup_{Y \subseteq X,\ Y \text{ finite}} \overline{Y} \qquad \text{and} \qquad \overline{X}^{\varepsilon} = \bigcup_{Y \subseteq X,\ Y \text{ finite}} \overline{Y}^{\varepsilon}.$$

By the general theory of closure operators, the closed sets of shapes under each of these closure operators form a complete algebraic lattice, with the operations of join and meet given respectively by

$$\begin{aligned} \bigwedge \overline{X}_i &= \bigcap \overline{X}_i,\ \bigvee \overline{X}_i = \bigwedge \left\{ \overline{X} : \bigcup X_i \subseteq X \right\} = \overline{\bigcup X_i} \\ \text{and } \bigwedge \overline{X}_i^{\varepsilon} &= \bigcap \overline{X}_i^{\varepsilon},\ \bigvee \overline{X}_i^{\varepsilon} = \bigwedge \left\{ \overline{X}^{\varepsilon} : \bigcup X_i^{\varepsilon} \subseteq X^{\varepsilon} \right\} = \overline{\bigcup X_i}^{\varepsilon}. \end{aligned}$$

Since there is an evident one-to-one correspondence between chapters and closed sets of shapes, we have a characterization of both **LC** and $\mathbf{LC}^{\varepsilon}$ in terms of sets of shapes. In particular equation (6) on page 12 follows.

Chapter 2 The Structure of $\mathbf{LC}^{\varepsilon}$

Section 2.1 Elementary Results about $\mathbf{LC}^{\varepsilon}$

We recall the following basic definitions from lattice theory:

- A lattice is called *complete* if every subset of the lattice has a meet and a join. An element a of a complete lattice L is called *compact* if and only if for every $X \subseteq L$ we have

  $$a \leq \bigvee X \Rightarrow (\exists \text{ finite } X_0 \subseteq X)(a \leq \bigvee X_0)$$

  A complete lattice is called *algebraic* if each of its elements is a join of compact elements.

- An element a of a lattice is called *join-irreducible* (JI) if and only if

  $$a = b \vee c \Rightarrow (a = b \text{ or } a = c).$$

- a is called *join-prime* (JP) if and only if

  $$a \leq b \vee c \rightarrow (a \leq b \text{ or } a \leq c).$$

- a is called *strictly join-irreducible* (SJI) if and only if

  $$a = \bigvee X \Rightarrow a \in X.$$

- a is called *strictly join-prime* (SJP) if and only if

  $$a \leq \bigvee X \Rightarrow \exists b \in X(a \leq b).$$

- The definitions of *meet-irreducible* (MI), *meet-prime* (MP), *strictly meet-irreducible* (SMI), and *strictly meet-prime* (SMP) are dual to these. The following relationships are immediate:

$$\mathrm{SJP} = (\text{compact} \cap \mathrm{JP}) \begin{array}{ccc} \subseteq & \mathrm{SJI} & \subseteq \\ & & \\ \subseteq & \mathrm{JP} & \subseteq \end{array} \mathrm{JI}$$

$$\mathrm{SMP} \begin{array}{ccc} \subseteq & \mathrm{SMI} & \subseteq \\ & & \\ \subseteq & \mathrm{MP} & \subseteq \end{array} \mathrm{MI}$$

- If $a < b$ and the open interval (a, b) is empty, then we say that b *covers* a, that b is an *upper cover* of a, and that a is a *lower cover* of b.

We recall also that if b is a compact element of a lattice and $a < b$, then b has a lower cover which is $\geq a$.

We are concerned to show that, like **LC**, the sublattice $\mathbf{LC}^{\epsilon}$ is distributive and algebraic, and also to show that important lattice-theoretic properties of elements of **LC** are preserved by the ε-mapping. We adopt the following terms and notations from [6]:

Definition 2.1.1

(i) A theory T and its chapter $|T|$ are called *connected* iff $|T| \in JI$.

(ii) C denotes the set of compact chapters of **LC**.

(iii) RE denotes the set of chapters containing a recursively enumerable theory.

(iv) CC denotes the set of compact and connected chapters.

(v) A chapter is called *normal* iff it lies below a compact chapter; N denotes the set of normal chapters and $\tilde{N}$ the set of non-normal chapters.

For clarity, we will distinguish meet and join on $\mathbf{LC}^{\varepsilon}$ by placing a small "ε" inside the usual symbols, e.g., "$\overset{\varepsilon}{\bigvee}$."

The following elementary result is proved for $\mathbf{LC}$ in [6]; we establish it here for the lattice $\mathbf{LC}^{\varepsilon}$. For any set of sentences A we will write $A^{\varepsilon} = (Th(A))^{\varepsilon}$, where $Th(A)$ denotes the first-order theory based on A.

Lemma 2.1.2

Let A_j be sets of axioms in disjoint languages. Then

$$\overset{\varepsilon}{\bigvee} \; |A_j^{\varepsilon}| = \left|\left(\bigcup A_j\right)^{\varepsilon}\right|.$$

Proof: Since evidently $|A_j^{\varepsilon}| \leq \left|\left(\bigcup A_j\right)^{\varepsilon}\right|$ for each j, we only need to show that $\left|\left(\bigcup A_j\right)^{\varepsilon}\right| \leq \overset{\varepsilon}{\bigvee} \; |A_j^{\varepsilon}|$. So suppose $\left(\bigcup A_j\right) \vdash_{\varepsilon} \sigma$ for some sentence σ, and let T be a representative of $\overset{\varepsilon}{\bigvee} \; |A_j^{\varepsilon}|$. Now σ is a consequence (in $\mathcal{L}^{\varepsilon}$) of finitely many of the A_j, say $\{A_{j_k}\}$, $k = 1, \ldots, n$. To show that T interprets σ, we note that since the A_j are assumed to be in disjoint languages, we have $\left|\left(\bigcup_k A_{j_k}\right)^{\varepsilon}\right| = \left|\left(A_{j_1} + \cdots + A_{j_n}\right)^{\varepsilon}\right|$. But it is clear that $\overset{\varepsilon}{\bigvee}_k \left|A_{j_k}^{\varepsilon}\right| = \left|\left(A_{j_1} + \cdots + A_{j_n}\right)^{\varepsilon}\right|$, and since $\overset{\varepsilon}{\bigvee}_k \left|A_{j_k}^{\varepsilon}\right| \leq \overset{\varepsilon}{\bigvee} \; \left|A_j^{\varepsilon}\right|$ we are done.

It is also shown in [6] that a chapter $|T|$ of $\mathbf{LC}$ is compact if and only if it has a representative that is finitely axiomatizable, i.e., there exists a sentence α such that $|T| = |\alpha|$. This is also true in $\mathbf{LC}^{\varepsilon}$:

Lemma 2.1.3

The following are equivalent:

i. $|T^{\varepsilon}|$ is compact in $\mathbf{LC}^{\varepsilon}$.

ii. There exists a sentence α such that $|\alpha^{\varepsilon}| = |T^{\varepsilon}|$.

iii. Every theory in $|T^{\varepsilon}|$ has a finitely axiomatizable sub-theory which also belongs to $|T^{\varepsilon}|$.

Proof: (ii) ⇔ (iii) is obvious.

(ii) ⇒ (i): Suppose there exists a sentence α such that $|\alpha^{\varepsilon}| = |T^{\varepsilon}|$, and suppose also that $|T^{\varepsilon}| \leq \bigvee^{\varepsilon} |T_j^{\varepsilon}|$. We may assume that the representatives T_j are chosen to be in disjoint languages. By Lemma 2.1.2, $|\alpha^{\varepsilon}| \leq |\left(\bigcup T_j\right)^{\varepsilon}|$, so $\left(\bigcup T_j\right) \vdash_{\varepsilon} \alpha^{I}$ for some interpretation I. Hence there is a finite set T_{j_k} such that $\left(\bigcup T_{j_k}\right) \vdash_{\varepsilon} \alpha^{I}$, whence $|T^{\varepsilon}| \leq \bigvee^{\varepsilon}{}_{k} |T_j^{\varepsilon}|$.

(i) ⇒ (iii): It is an easy fact that for any theory T,

$$|T^{\varepsilon}| = \bigvee^{\varepsilon} \{|\sigma^{\varepsilon}| : T \vdash_{\varepsilon} \sigma\}, \tag{7}$$

for every theorem on one side is evidently interpretable in the other side. Now if $|T^{\varepsilon}|$ is compact then there is a finite set $\{\sigma_k\}_{k=1,\ldots,n}$ such that $|T^{\varepsilon}| = \bigvee^{\varepsilon}{}_{k} |\sigma_k^{\varepsilon}|$, hence $|T^{\varepsilon}| = |\{\sigma_1, \ldots, \sigma_n\}^{\varepsilon}|$.

Corollary 2.1.4

If $|T|$ is compact in **LC** then $|T^{\varepsilon}|$ is compact in $\mathbf{LC}^{\varepsilon}$.

Proof: Suppose $|T|$ is compact in **LC**, and let be α be a sentence such that $|\alpha| = |T|$. Then $|\alpha^{\varepsilon}| = |T^{\varepsilon}|$, so by the above lemma, $|T^{\varepsilon}|$ is compact.

We have already established (Theorem 1.3.1) that $\mathbf{LC}^{\varepsilon}$ is a complete, distributive algebraic lattice. We have then also the following corollary.

Corollary 2.1.5

$\mathbf{LC}^\varepsilon$ satisfies the Brouwerian law, i.e., for any $a \in \mathbf{LC}^\varepsilon$ and $B \subseteq \mathbf{LC}^\varepsilon$,

$$a \underset{\varepsilon}{\wedge} \bigvee^{\varepsilon} B = \bigvee^{\varepsilon} \{a \underset{\varepsilon}{\wedge} b \mid b \in B\},$$

and in $\mathbf{LC}^\varepsilon$ (as in $\mathbf{LC}$) we have $JI = JP$, $MI = MP$, and $SJI = SJP = CC$.

> *Proof*: The Brouwerian law holds in every distributive, algebraic lattice, $SJI = SJP$ follows from the Brouwerian law, and the other equalities follow from distributivity.

Theorem 2.1.6

(i) $|T|$ is compact in $\mathbf{LC} \Rightarrow |T^\varepsilon|$ is compact in $\mathbf{LC}^\varepsilon$.

(ii) $|T| \in JP$ (in $\mathbf{LC}$) $\Rightarrow |T^\varepsilon| \in JP$ (in $\mathbf{LC}^\varepsilon$).

(iii) $|T| \in JI$ (in $\mathbf{LC}$) $\Rightarrow |T^\varepsilon| \in JI$ (in $\mathbf{LC}^\varepsilon$).

(iv) $|T| \in CC$ (in $\mathbf{LC}$) $\Rightarrow |T^\varepsilon| \in CC$ (in $\mathbf{LC}^\varepsilon$).

> *Proof*: (i) Suppose $|T|$ is compact in $\mathbf{LC}$, and let be α be a sentence such that $|\alpha| = |T|$. Then $|\alpha^\varepsilon| = |T^\varepsilon|$, so by Lemma 2.1.7, $|T^\varepsilon|$ is compact.
>
> (ii) Let $|T|$ be join prime in $\mathbf{LC}$, and suppose $|T^\varepsilon| \leq \bigvee X$, $X \subseteq \mathbf{LC}^\epsilon$. Then in $\mathbf{LC}$ we must have $|T| \leq \bigvee X$, so by join primeness there is an $|S| \in X$ such that $|T| \leq |S|$, whence $|T^\epsilon| \leq |S^\epsilon|$ by Corollary 1.2.7. But since $|S| \in X$ we have $|S^\epsilon| = |S|$, so $|T^\epsilon| \leq |S|$ and we are done.
>
> (iii) Follows from (ii) and the fact that $JI = JP$ in $\mathbf{LC}^\varepsilon$.
>
> (iv) Follows by definition from (i) and (ii).

It is shown in [6] that $|\mathbf{LC}| = 2^{\aleph_0}$; using an auxiliary result of [6] we establish below that $|\mathbf{LC}^{\varepsilon}|$ is also of size continuum.

Theorem 2.1.7

$Card\ (\mathbf{LC}^{\varepsilon}) = 2^{\aleph_0}$.

Proof: Since the elements of $\mathbf{LC}^{\varepsilon}$ (and $\mathbf{LC}$) can be represented by sets of shapes and there are only countably many shapes, we know that $Card\ (\mathbf{LC}^{\varepsilon}) \leq 2^{\aleph_0}$.

Now we establish the converse inequality. Recall that Peano's Arithmetic **PA** is a theory with definable selectors. The same is true for every extension of **PA** (in the same language). Hence if T is such an extension, then $T^{\varepsilon} \preceq T$, i.e., $|T^{\varepsilon}| = |T|$. Using a theorem of R. Montague it is shown in [6] that there exists a sequence $\{T_n\}_{n=1,2,\ldots}$ of extensions of **PA** such that

$$|T_i| \not\preceq \left| \bigcup_{j \neq i} T_j \right|,$$

for $i = 1, 2, \ldots$. Let **A** be a set of $2^{\aleph_0}$ almost disjoint subsets of $\{1, 2, \ldots\}$. Thus the chapters $\left| \bigcup_{j \in S} T_j \right|$, $S \in \mathbf{A}$, constitute an antichain in $\mathbf{LC}^{\varepsilon}$ which is of power $2^{\aleph_0}$.

This concludes our remarks about $\mathbf{LC}^{\varepsilon}$. It would be interesting to review [6] more completely and establish which of the other results of that paper are true for $\mathbf{LC}^{\varepsilon}$.

Chapter 3 The Syntax of Hologic

Section 3.1 Introduction

We now introduce Hologic[1], a formal predicate logic that is motivated by the conceptual basis of Hilbert's ε-operator, but differs essentially from the ε-extension as it is classically understood.

The ε-operator replaces variables with "indeterminate constants." That is, the term $\varepsilon x \varphi$ occuring in the sentence $\varphi(\varepsilon x \varphi)$, when interpreted, refers to some element of the universe, but an element about which we can make only a very limited claim, namely that if there is *any* element that satisfies φ, then $\varepsilon x \varphi$ is such an element. Aside from this characteristic, the element referenced by $\varepsilon x \varphi$ remains completely *un*determined. Supposing there is more than one element that satisfies φ, we do not know which one is represented by $\varepsilon x \varphi$. Moreover, if no element satisfies φ, then $\varepsilon x \varphi$ refers to an element about which we know nothing at all. [5]

One might say that the ε-operator, rather than representing a *choice* (of witness) as a Skolem function does, represents the logician's *capacity* to choose, his agency as an interpreter and user of the language. The pursuit of a formalism that codifies this quality of the ε-operator has led to the predicate logic that we will describe herein.

Hologic is characterized by two distinct features. The second in order of presentation is the first in importance: Hologic is built on a radical generalization of quantificational semantics. Classically, predicates are interpreted as relations on a set, and the semantics of quantification reduces to the properties of such relations. The semantics of Hologic, while including such set-theoretic interpretations as a special case, permits quantification to be interpreted operationally, i.e., as one or more claims about what is

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[1] Coined from **hol-**, the combining form of the Greek ὅλος, meaning "whole, entire," and **logic**, from the Greek $\lambda o \gamma \iota \kappa \eta$, whence the Latin *logicus*, "pertaining to reasoning."

determinable, and moreover permits such claims to be interpreted over domains that are not completely specified. This generalization is non-trivial, in that we will exhibit sentences that are classically valid but Hologically invalid. This is expected — in the shift from “what can I suppose to exist?” to “what can I suppose that I may actually lay my hands on?” the truth conditions for any predicate sentence will become if anything more stringent, not less.

The other characteristic feature of Hologic is that both quantification and negation are syntactically *infix*. While not strictly necessary, in the sense that we could have fitted the new semantics to the old (prefix) notation, it was found to be much simpler and clearer to take the approach that we exposit below.

We are also motivated by our interest in making Hologic more amenable to computational treatment. Despite a half-century of effort the means of constructing a recognizably thoughtful machine – the Holy Grail of practical computer science – eludes us. Yet during that half-century the speed and complexity of computing machinery have advanced so extravagantly that any practical physical limit to computation now seems an unlikely cause of this failure, and the art and science of programming has advanced so far as to challenge the limits of humanly comprehensible complexity. We are forced to conclude that our primary mental faculties are too complex for human understanding – or else that we are missing something. And if we are missing something, and the something is not unknowable, then it may well be something to do with our approach to logic.

We do not claim that Hologic is the answer. However, it seems to us that it possesses certain relevant qualities. It frees us from thinking of logic as a game that consists solely in rules for manipulating symbol strings, and returns us instead to thinking of logic more naturally, as existing in and emerging from the relationships among properties (predicates) and the world-objects to which properties appertain. The semantics of Hologic neither assumes nor demands perfect information about a domain, nor does it require that quantificational relationships be order-dependent. (See in particular Remark 5.2.4.) Syntactically Hologic has a recursive structure that is hierarchical and modular, so that substitutions and inference rules may be applied easily and elegantly. Expressions are most naturally represented not as strings of symbols but as computational objects whose structure may be captured in most cases by a few bytes' worth of pointers. Powerful new computational methods for both theorem checking and theorem searching may be attainable as a consequence.

We will see that every sentence of Hologic is required to be in a canonical form, called Nested Normal Form (NNF). Consequently, before presenting the language, it will be worthwhile to state in this introductory section a technical result about propositional sentences, which we call the nesting lemma. In what follows we denote propositional sentences by calligraphic capital letters such as $\mathcal{A}$, $\mathcal{B}$, etc., and propositional variables by Roman capitals. Tautological equivalence will be denoted by "$\equiv$". We write expressions such as $\mathcal{A}(A_1, \ldots, A_n)$ to indicate that the sentence $\mathcal{A}$ consists of the propositional atoms (variables) $A_1, \ldots, A_n$ joined in some manner by logical connectives. We will say that a sentence $\mathcal{A}(A_1, \ldots, A_n)$ is *concise* if it is not tautologically equivalent to a sentence in which not all of the variables $A_1, \ldots, A_n$ appear.

Definition 3.1.1 (Nests and Nesting)

Let $\mathcal{G}(A_1, \ldots, A_l, B_1, \ldots, B_m, C_1, \ldots, C_n)$ denote a concise sentential formula in the (arbitrarily labeled) propositional variables $A_1, \ldots, A_l$, $B_1, \ldots, B_m$, and $C_1, \ldots, C_n$, with $l > 0$, $m > 0$, and $n \geq 0$. If $\mathcal{G}$ may be written as

$$\mathcal{G} \equiv \mathcal{A}(A_1, \ldots, A_l, C_1, \ldots, C_n, \mathcal{B}(B_1, \ldots, B_m, C_1, \ldots, C_n)),$$

then we will say that $\mathcal{G}$ is *nested*, that $\mathcal{A}$ is a *nesting* of $\mathcal{G}$, that $\mathcal{B}$ is the corresponding *nest*, and that this nest *captures the B's*. We will also say that $\{B_1, \ldots, B_m, C_1, \ldots, C_n\}$ is a *nestable set for $\mathcal{G}$ that captures the B's*, and we will call $\{C_1, \ldots, C_n\}$ the set of *common variables* for $\mathcal{A}$ and $\mathcal{B}$. If there are no common variables (i.e., $n = 0$), then we will say that $\mathcal{G}$ is *strictly nested* on the B's.

The nesting of sentential formulas[2] occurs naturally in discourse; it is a way of forming simpler sentences out of complicated ones by "chunking up" one or more of the concepts expressed within the sentence into an atomic expression, or into what might be called a complex propositional variable. For example, the set $\{A_2, A_3\}$ is strictly nestable for the sentence $(A_1 \wedge A_2) \vee (A_1 \wedge A_3)$ since this sentence may be rewritten as the tautologically equivalent sentence $A_1 \wedge \mathcal{B}(A_2, A_3)$, where $\mathcal{B}(A_2, A_3) = (A_2 \vee A_3)$. All propositional atoms are trivially nested, but a nesting that captures a set of two or more atoms can fail to exist for a given sentence. For instance, the set $\{A_1, A_3\}$ cannot be captured by any proper nesting of $(A_1 \wedge A_2) \vee (A_1 \wedge A_3)$. Since there are only finitely many Boolean functions of n variables (namely 2^{2^n}) the following lemma is obvious.

Lemma 3.1.2 (Nesting Lemma)

Let $\mathcal{G}(A_1, \ldots, A_l, B_1, \ldots, B_m, C_1, \ldots, C_n)$ denote a sentential formula. Then there is an effective procedure for determining whether $\{B_1, \ldots, B_m, C_1, \ldots, C_n\}$ is a nestable set for $\mathcal{G}$ that captures the B's.

For specific algorithms see [1] and [2].

---

[2] Nestings, also called Boolean function decompositions, were studied at Bell Labs, where the aim was to make the design of switching circuits as efficient as possible. The principle results, including our "nesting lemma," were first published by R.L. Ashenhurst (at Bell Labs) in 1952.

Section 3.2 Flat Sentences

Definition 3.2.1 (Symbols)

We use the following sets of symbols:

- *Basic symbols* $v_0, v_1, \ldots$. We use natural number subscripts to emphasize that there are infinitely many of these. In practice we use x, y, $z, \ldots$.
- The diacritical marks $\check{}$ and $\hat{}$ (called "check" and "hat"), and the overline $\overline{}$.
- For each non-negative integer n a set of n-place predicate symbols $P^n_0, P^n_1, \ldots$. In practice we use P, $Q, \ldots$ and say what number of places they have.
- The logical connectives $\wedge$ and $\vee$.

Definition 3.2.2 (Indeterminates and Rooted Sets of Indeterminates)

An *indeterminate* is an expression of the form $\check{x}^A$ or $\hat{x}^A$, where A is a finite set (possibly empty) of basic symbols different from x. The basic symbol x is called the *root symbol* of the indeterminate, and A is called the *argument* of the indeterminate. A checked indeterminate, i.e., an indeterminate of the form $\check{x}^A$, is call a *witness*. A hatted indeterminate, i.e., an indeterminate of the form $\hat{x}^A$, is called a *falsifier*. The indeterminate $\check{x}^A$ is called the *dual* of the indeterminate $\hat{x}^A$, and *vice versa*. That is, two indeterminates are dual if one is a witness and the other a falsifier, but they have the same root symbol and the same argument. When we wish to denote an arbitrary indeterminate (i.e., without specifying whether it is a witness or a falsifier) we may use a dot in place of a check or a hat, e.g., $\dot{x}^A$. We will say that a set of indeterminates is *rooted* if every basic symbol occurring among the indeterminates in the set has exactly one root occurrence.

Definition 3.2.3 (The Dependency Relation)

Let $I = \{\dot{x}_1^{A_1}, \ldots, \dot{x}_n^{A_n}\}$ be a rooted set of indeterminates. Let S_I denote the set of basic symbols occurring in I. We associate with I a relation $\mathbf{R}_I$ on the set S_I given by $\mathbf{R}_I = \{(x_i, x_j) \mid \dot{x}_i^{A_i} \in I \text{ and } x_j \in A_i\}$. This is called the *dependency relation* for the set of indeterminates I. If we say a set of indeterminates I has some specific ordering, we mean that the dependency relation is the specified ordering. If $\dot{x}_i^{A_i}, \dot{x}_j^{A_j} \in I$ then we will say that $\dot{x}_i^{A_i}$ *depends upon* $\dot{x}_j^{A_j}$ if and only if $(x_i, x_j) \in \mathbf{R}_I$, and we will write $\dot{x}_i^{A_i} \prec \dot{x}_j^{A_j}$ to denote this.

Definition 3.2.4 (Free Indeterminates and Trifles)

Let $\dot{x}^A$ be an indeterminate in a set of indeterminates I. If $\dot{x}^A$ depends on no other indeterminate occurring in I, then $\dot{x}^A$ is said to be *free* for I. Conversely, if no other indeterminate occurring in I depends on $\dot{x}^A$, then $\dot{x}^A$ is said to be a *trifle* in I.

Definition 3.2.5 (Predicate Complements)

To each n-place predicate symbol P we associate the expression $\overline{P}$, which we also consider to be an n-place predicate symbol. We call $\overline{P}$ the *complement* of P and *vice versa*.

We will see later that when interpreted $\overline{\overline{P}}$ is always the same as P.

Definition 3.2.6 (Literals and Flat Sentences)

1. Let P be an n-place predicate symbol, let $\tau_0, \dots, \tau_{n-1}$ be a sequence of indeterminates, and let $I_P = \{\dot{x}_1^{A_1}, \dots, \dot{x}_k^{A_k}\}_{1 \leq k \leq n}$ be the set of indeterminates occurring in the sequence $\tau_0, \dots, \tau_{n-1}$. If the members of I_P are linearly-ordered by the dependency relation on I_P, then the expression $P\tau_0, \dots, \tau_{n-1}$ is called an *atomic literal*.
2. A disjunction of literals is called a *clause*, and a conjunction of clauses is called a *presentence in conjunctive normal form*. If a presentence in conjunctive normal form is rewritten, using the standard rewrite rules of Boolean logic, as a disjunction of conjunctions of literals, the result is a *presentence in disjunctive normal form*.
3. Let φ be a presentence. If the set I_φ of indeterminates of φ is rooted and linearly-ordered, then φ is called a *flat sentence*. A flat sentence consisting of a single atomic literal is called an atomic sentence.

We require flat sentences to be in a disjunctive or conjunctive normal form to simplify the development of the semantics of Hologic in the next chapter. To simplify our presentation, specifically to avoid writing the self-evident duals to each subsequent definition or proposition involving flat sentences, we will assume that any arbitrary flat sentence is in the conjunctive form unless otherwise stated.

Section 3.3 Complex Literals and Sentences

A fundamental syntactical feature of Hologic is that under certain conditions a given sentence may take on the rôle of a literal — what we term a *complex literal* — as a nest within another sentence. (“Nest” is used here in the sense of Definition 3.1.1.) For convenience we will use special predicate symbols to denote complex literals. These special symbols will be set in calligraphic font, e.g., $\mathcal{P}$ for P, $\mathcal{Q}$ for Q, etc., and we will call them *complex literal symbols*. For a sentence to become a complex literal in another sentence it is necessary that the sets of indeterminates of both sentences be in some sense compatible. For instance, we could not have a witness and a falsifier appearing in the same sentence with the same basic root symbol, or have a closed chain of dependency among the indeterminates. To prevent such problems, we first need to draw a distinction between what we term "local" and "non-local" indeterminates. We then define what it means for two sets of indeterminates to be comparable.

> Definition 3.3.1 (Local and Non-Local Indeterminates)
>
> Suppose φ is an expression consisting of literals joined in some manner by the logical connectives $\vee$ and $\wedge$. An indeterminate $\dot{x}^{\mathcal{A}}$ of φ occurring in a literal $P\tau_0, \ldots, \tau_{n-1}$ is said to be *local to* $P\tau_0, \ldots, \tau_{n-1}$ if it does not occur in any literal that is distinct from $P\tau_0, \ldots, \tau_{n-1}$ and if its root symbol does not occur in the argument of any indeterminate with an occurrence in a literal distinct from $P\tau_0, \ldots, \tau_{n-1}$. Otherwise $\dot{x}^{\mathcal{A}}$ is said to be *non-local* to $P\tau_0, \ldots, \tau_{n-1}$.
>
> Definition 3.3.2 (Comparable Sets of Indeterminates and Expansion)
>
> Two sets of indeterminates I and J are said to be *comparable* if they have the same root symbols and if no root symbol is the root of a witness in one set and the root of a falsifier in the other set. (I.e., the hats/checks correspond for corresponding symbols.) If in addition the argument to each indeterminate in J contains the argument to each indeterminate in I having the same root symbol, then we say that J is an *expansion* of I.

We are now in a position to give the general definition of "sentence" for Hologic.

Definition 3.3.3 (Sentences)

> Every flat sentence is a sentence. Suppose φ and ψ are sentences such that for some literal $P\tau_0, \ldots, \tau_{n-1}$ occurring in ψ the set I_P of all the indeterminates occurring in $P\tau_0, \ldots, \tau_{n-1}$ is an expansion of the set I_φ of non-local indeterminates occurring in φ. Then we may form a new expression ψ' by substituting φ for (one or more occurrences) of $P\tau_0, \ldots, \tau_{n-1}$ in ψ, replacing each indeterminate in I_φ by the corresponding indeterminate from I_P. If φ is not an atomic sentence, then each indicated occurrence of φ (modified as above) in ψ' will be called a *nest* or a *complex literal* in ψ', and we will say that ψ' is *a sentence in nested normal form* (NNF). We will also in this case use the special grouping symbols "$\lfloor \ldots \rfloor$" or "$\lceil \ldots \rceil$" to demarcate the nest.

As noted previously, we may for convenience use a complex predicate symbol to represent the nest. In general this gives us many distinct ways of writing a sentence, depending on whether we choose to use complex predicate symbols, and which complex literals we choose to use them for. The following example is intended to illustrate both the definition of "sentence" and the conventions which in practice we will adopt for writing sentences.

For example, let

$$\psi = P\check{u}\hat{v}^{u}\check{w}^{uv} \wedge (Q\hat{v}^{u}\check{u}\check{w}^{uv} \vee R\check{w}^{uv}\check{u}\hat{x}^{uw}),$$

$$\varphi = S\check{u}\check{w}^{u}\hat{x}^{uw} \wedge T\check{u}\check{w}^{u}\hat{x}^{uw}\check{y}^{uwx}.$$

Then $I_R = \{\check{u}, \check{w}^{uv}, \hat{x}^{uw}\}$ is an expansion of the set $I_\varphi = \{\check{u}, \check{w}^{u}, \hat{x}^{uw}\}$ of non-local indeterminates in φ. Consequently, setting

$$\mathcal{R}\check{w}^{uv}\check{u}\hat{x}^{uw} = S\check{u}\check{w}^{uv}\hat{x}^{uw} \wedge T\check{u}\check{w}^{uv}\hat{x}^{uw}\check{y}^{uwx}$$

we may write the new sentence

$$\psi' = P\check{u}\hat{v}^{u}\check{w}^{uv} \wedge (Q\hat{v}^{u}\check{u}\check{w}^{uv} \vee \mathcal{R}\check{w}^{uv}\check{u}\hat{x}^{uw})$$

with complex literal $\mathcal{R}\check{w}^{uv}\check{u}\hat{x}^{uw}$. Note that the local indeterminate $\check{y}^{uwx}$ does not occur as an argument to $\mathcal{R}$. However, we can also write ψ' as

$$\psi' = P\check{u}\hat{v}^{u}\check{w}^{uv} \wedge (Q\hat{v}^{u}\check{u}\check{w}^{uv} \vee \lfloor S\check{u}\check{w}^{uv}\hat{x}^{uw} \wedge T\check{u}\check{w}^{uv}\hat{x}^{uw}\check{y}^{uwx}\rfloor)$$

where the grouping symbols "$\lfloor$" and "$\rfloor$" are used to demarcate the nest, and we call this an *expanded display* of ψ'. (For expanded display of nests within nests, we alternate using the grouping symbols "$\lfloor \ldots \rfloor$" and "$\lceil \ldots \rceil$".)

When choosing labels (predicate symbols) for complex literals, it may happen that two complex literals occurring in a sentence may be so much alike — though not identical — that it is appropriate to label them with the same complex predicate symbol. The following definition makes this precise.

> Definition 3.3.4 (Alike Literals)
>
> Any two complex literals occurring in a sentence are said to be *alike literals* if they are identical up to the choice of conjunctive or disjunctive form (and the order of the conjuncts/disjuncts) for the nest and the labeling of the local indeterminates, and except for the labeling, polarity, and order of dependence among the non-local indeterminates. Complex literals are said to be *alike-complements* if each is alike to the negation of the other.

We will label alike literals with the same complex predicate symbol, and alike-complement literals with complementary predicate symbols. Complex literals may then be identified or distinguished on the basis of their predicate labels and arguments, just as atomic literals are.

For example, the following three complex literals maybe distinguished as described below:

$$\begin{aligned} \mathcal{P}_1\check{x} &= P\check{x}\hat{y}^x \vee Q\check{z}^x\check{x} \\ \mathcal{P}_1\hat{x} &= P\hat{x}\hat{y}^x \vee Q\check{z}^x\hat{x} \\ \mathcal{P}_2\check{x} &= P\check{x}\check{y}^x \vee Q\check{z}^x\check{x} \end{aligned}$$

The first two are distinguished by having different arguments (a witness in the one case, a falsifier in the other). Also, the first two are distinguished from the third by the predicate labels, and the labels are different because the indeterminate local to P is a falsifier in the expanded form of $\mathcal{P}_1$ and a witness in the expanded form of $\mathcal{P}_2$.

The two complex literals below may be identified with one another. They have the same argument, and they have the same label because the expanded forms are identical up to the order of the disjunction and the labeling of the local indeterminates:

$$\begin{aligned} \mathcal{P}_1\check{x} &= P\check{x}\hat{y}^x \vee Q\check{z}^x\check{x} \\ \mathcal{P}_1\check{x} &= Q\check{v}^x\check{x} \vee P\check{x}\hat{u}^x \end{aligned}$$

The way in which we have defined complex literals and the nested normal form permits sentences that are already nested to be used as complex literals in turn. This leads us to the notion of sentence depth:

> Definition 3.3.5 (Depth)
>
> The depth of an atomic literal is 0, and the depth of a complex literal is 1 greater than the depth of the deepest literal nested within it. The depth of a sentence is 1 greater than the depth of its deepest nest.

Thus atomic sentences have depth 0, flat sentences have depth 1, the example sentence ψ' on page 28 has depth 2, and so on.

Definition 3.3.6 (Negation)

> We define a syntactic operation, which we call *negation*. Let $P\tau_0, \ldots, \tau_{n-1}$ be an atomic literal occurring in a sentence φ. To negate an instance of $P\tau_0, \ldots, \tau_{n-1}$ within φ we replace that instance's predicate symbol with the it's complement symbol $\overline{P}$, and replace each *local* indeterminate with its dual. (We may sometimes denote this by $\overline{P\tau_0, \ldots, \tau_{n-1}}$.) If φ is a flat sentence, it is negated by negating each literal occurring in it, replacing each conjunction with a disjunction and each disjunction with a conjunction, and flipping the non-local indeterminates as well. In general, a sentence is negated by recursion on its depth.

For instance, to negate the complex literal $\mathcal{R}\check{w}^{uv}\check{u}\hat{x}^{uw}$ in the example sentence

$$\psi' = P\check{u}\hat{v}^{u}\check{w}^{uv} \wedge (Q\hat{v}^{u}\check{u}\check{w}^{uv} \vee \mathcal{R}\check{w}^{uv}\check{u}\hat{x}^{uw})$$

to obtain a new sentence ψ'' we would write

$$\psi'' = P\check{u}\hat{v}^{u}\check{w}^{uv} \wedge (Q\hat{v}^{u}\check{u}\check{w}^{uv} \vee \overline{\mathcal{R}}\check{w}^{uv}\check{u}\check{x}^{uw}).$$

Note that the local indeterminate $\hat{x}^{uw}$ was flipped to $\check{x}^{uw}$. The expanded display of this sentence shows how the negation was carried through to the binary connector and each constituent literal occurring in the nest:

$$\psi'' = P\check{u}\hat{v}^{u}\check{w}^{uv} \wedge (Q\hat{v}^{u}\check{u}\check{w}^{uv} \vee \lfloor \overline{S}\check{u}\check{w}^{uv}\check{x}^{uw} \vee \overline{T}\check{u}\check{w}^{uv}\check{x}^{uw}\check{y}^{uwx} \rfloor).$$

On the other hand, to negate the entire sentence ψ' we would write

$$\overline{\psi'} = \overline{P}\hat{u}\check{v}^{u}\hat{w}^{uv} \vee (\overline{Q}\check{v}^{u}\hat{u}\hat{w}^{uv} \wedge \overline{\mathcal{R}}\hat{w}^{uv}\hat{u}\check{x}^{uw}).$$

Section 3.4 Formulas and Forms

The word “formula” has been used in mathematical logic to denote any well-formed expression, especially one in which not all of the variables are bound by a quantifier. However, in Hologic there are no variables, free or otherwise: every well-formed expression is a sentence. Therefore we will use the word “formula” in quite a different manner, but one that we believe makes appropriate use of its historical sense.

> Definition 3.4.1 (Equivalent Sets of Indeterminates)
>
> Let I be a rooted set of indeterminates, and suppose that either $\check{x}$ or $\hat{x}$ occurs in I, but that neither $\check{y}$ nor $\hat{y}$ occurs in I. We denote by $I^{x \leftarrow y}$ the set of indeterminates obtained from I by replacing each (root and non-root) instance of the basic symbol x in I by y. If I and J are sets of indeterminates and there exist sets of indeterminates $I_1, I_2, \ldots, I_n$ such that $J = I_n^{x_n \leftarrow y_n}, \ldots, I_2 = I_1^{x_1 \leftarrow y_1}, I_1 = I^{x_0 \leftarrow y_0}$ then we shall say that I and J are *equivalent* sets, denoted $I \simeq J$.

In other words, two sets of indeterminates are equivalent when one can be obtained from the other by a uniform substitution of indeterminate symbols, keeping distinct symbols distinct, introducing no new distinctions, and preserving hats and checks.

> Definition 3.4.2 (Formulas)
>
> Let φ_I denote a sentence with set of indeterminates I. If $I \simeq J$, we denote by φ_J^I the sentence obtained by replacing each indeterminate from I appearing in φ with the corresponding indeterminate from J. If φ_I and ψ_J are sentences, $I \simeq J$, and $\psi_J = \varphi_J^I$, then we will say that φ_I and ψ_J are *isomorphic*, denoted $\varphi_I \simeq \psi_J$. We observe that $\simeq$ is an equivalence relation on sentences. If φ_I is a sentence, we define the *formula* φ to be the equivalence class $\{\psi_J : \psi_J = \varphi_J^I, J \simeq I\}$.

One further abstraction is appropriate:

> Definition 3.4.3 (Forms)
>
> Given a formula φ, we define the *form* of φ to be the equivalence class of formulas that can be obtained from φ by uniformly relabeling the predicate symbols, keeping distinct symbols distinct and complementary symbols complementary.

We end our development of the syntax of Hologic with the observation that the only distinction to be made between sentences of Hologic and complex literals is a distinction of context; a sentence is a complex literal standing alone. We will now turn to the semantics of Hologic.

Chapter 4 The Semantics of Hologic

Section 4.1 Strategies and Strategic Assignment

Definition 4.1.1 (Universes of Discourse and Predicates)

By a *universe of discourse* we mean a non-empty set or class $\mathbb{U}$ of objects. We denote by $\mathscr{P}(\mathbb{U})$ the class of sets of objects from $\mathbb{U}$. By an n-place *predicate* on $\mathbb{U}$ we mean a function whose domain is one or more n-tuples of objects from $\mathbb{U}$ and whose range is contained in the set $\{true, false\}$. We denote a predicate on $\mathbb{U}$ by a predicate symbol with a superscript: $P^{\mathbb{U}}$. The set of n-tuples forming the domain of $P^{\mathbb{U}}$ is denoted $dom\left(P^{\mathbb{U}}\right)$. To each predicate $P^{\mathbb{U}}$ we associate a predicate $\overline{P}^{\mathbb{U}}$ having the same domain and such that, for every $< a_0, \ldots, a_{n-1} > \in dom\left(P^{\mathbb{U}}\right)$, $\overline{P}^{\mathbb{U}} a_0 \ldots a_{n-1} = true \Leftrightarrow P^{\mathbb{U}} a_0 \ldots a_{n-1} = false$.

Definition 4.1.2 (Knowledge Functions and Sets, Comprehension)

We treat each positive integer n as an ordinal, i.e., $n = \{0, 1, \ldots, n-1\}$. Let $\alpha \subseteq n$, and let $i = |\alpha|$. Let $\mathbb{U}$ be a universe of discourse, and let $P^{\mathbb{U}}$ be an n-place predicate on $\mathbb{U}$. If $< a_0, \ldots, a_{n-1} > \in dom\left(P^{\mathbb{U}}\right)$ then let $\pi_\alpha < a_0, \ldots, a_{n-1} >$ denote the i-tuple that is the projection of $< a_0, \ldots, a_{n-1} >$ onto just those coordinates indexed by elements of α, and let $\pi_\alpha\left(dom\left(P^{\mathbb{U}}\right)\right)$ be the set of all such projections of elements of $dom\left(P^{\mathbb{U}}\right)$.

Now let β be a non-empty subset of the complement of α in n, i.e., $\beta \subseteq n \setminus \alpha$ and $\beta \neq \emptyset$. A *knowledge function* for the pair (α, β) on $P^{\mathbb{U}}$ is a function $\gamma_{<\alpha,\beta,P,\mathbb{U}>}$ whose domain is $\pi_\alpha\left(dom\left(P^{\mathbb{U}}\right)\right)$ and whose range is $\mathscr{P}(\mathbb{U})$. In other words, a knowledge function on $P^{\mathbb{U}}$ indexed by the pair

(α, β) maps the projection π_α of each element of the domain of $P^{\mathbb{U}}$ to a set of objects in the universe of discourse. Such sets of objects are called *knowledge sets.* Knowledge sets may be empty.

A collection $\mathit{\Gamma}_{P^{\mathbb{U}}}$ of knowledge functions for a given n-place predicate $P^{\mathbb{U}}$that consists of exactly one knowledge function $\gamma_{<\alpha,\beta,P,\mathbb{U}>}$ for each pair (α, β) such that $\alpha \subseteq n$, $\beta \subseteq n \backslash \alpha$, and $\beta \neq \emptyset$, is called a *comprehension* of $P^{\mathbb{U}}$.

We observe that if $P^{\mathbb{U}}$ is an n-place predicate, then a comprehension of $P^{\mathbb{U}}$ consists of exactly $3^n - 2^n$ functions. This is because the number of distinct subsets α of n of size i is $\binom{n}{i}$, and for each of these there are $2^{n-i} - 1$ choices for β. We have

$$\begin{aligned}
|\mathit{\Gamma}| &= \sum_{i=0}^{n} \binom{n}{i} \left(2^{n-i} - 1\right) \\
&= \sum_{i=0}^{n} \binom{n}{i} 2^{n-i} - \sum_{i=0}^{n} \binom{n}{i} \\
&= (1+2)^n - (1+1)^n \qquad \text{(Binomial Theorem)} \\
&= 3^n - 2^n.
\end{aligned}$$

We also observe that any comprehension of $P^{\mathbb{U}}$ is a comprehension of $\overline{P}^{\mathbb{U}}$.

Definition 4.1.3 (Knowledge Properties)

CUMULATIVE: A comprehension $\mathit{\Gamma}_{P^{\mathbb{U}}}$ of an n-place predicate $P^{\mathbb{U}}$ is said to have the *cumulative knowledge property* if, for any two subsets α and α' of n with $\alpha \subseteq \alpha'$, and for β a subset of both $n \backslash \alpha$ and $n \backslash \alpha'$, we have

$$\gamma_{<\alpha,\beta,P,\mathbb{U}>} (\pi_\alpha < a_0, \ldots, a_{n-1} >) \subseteq \gamma_{<\alpha',\beta,P,\mathbb{U}>} (\pi_{\alpha'} < a_0, \ldots, a_{n-1} >)$$

for all $< a_0, \ldots, a_{n-1} > \in dom\left(P^{\mathbb{U}}\right)$ and all $\gamma \in \mathit{\Gamma}_{P^{\mathbb{U}}}$.

THEORETIC: A comprehension $\varGamma_{P^{\mathbb{U}}}$ of an n-place predicate $P^{\mathbb{U}}$ is said to have the *theoretic knowledge property* if, for any two subsets α and α' of n and for β a subset of both $n \backslash \alpha$ and $n \backslash \alpha'$, we have

$$\gamma_{<\alpha,\beta,P,\mathbb{U}>}(\pi_{\alpha} < a_0, \ldots, a_{n-1} >) = \gamma_{<\alpha',\beta,P,\mathbb{U}>}(\pi_{\alpha'} < a_0, \ldots, a_{n-1} >)$$

for all $< a_0, \ldots, a_{n-1} > \in dom\left(P^{\mathbb{U}}\right)$ and all $\gamma \in \varGamma_{P^{\mathbb{U}}}$.

SET-THEORETIC: A comprehension $\varGamma_{P^{\mathbb{U}}}$ of an n-place predicate $P^{\mathbb{U}}$ is said to have the *set-theoretic knowledge property* if there exists a set $A \subseteq \mathbb{U}$ such that for every α and β we have

$$\gamma_{<\alpha,\beta,P,\mathbb{U}>}(\pi_{\alpha} < a_0, \ldots, a_{n-1} >) = A$$

for all $< a_0, \ldots, a_{n-1} > \in dom\left(P^{\mathbb{U}}\right)$ and all $\gamma \in \varGamma_{P^{\mathbb{U}}}$.

UNIVERSAL: A comprehension $\varGamma_{P^{\mathbb{U}}}$ of an n-place predicate $P^{\mathbb{U}}$ is said to have the *universal knowledge property* if the universe of discourse $\mathbb{U}$ is a set and if for every α and β we have

$$\gamma_{<\alpha,\beta,P,\mathbb{U}>}(\pi_{\alpha} < a_0, \ldots, a_{n-1} >) = \mathbb{U}$$

for all $< a_0, \ldots, a_{n-1} > \in dom\left(P^{\mathbb{U}}\right)$ and all $\gamma \in \varGamma_{P^{\mathbb{U}}}$.

We observe that the knowledge properties form a hierarchy, in that universal comprehensions are set-theoretic, set-theoretic comprehensions are theoretic, and theoretic comprehensions are cumulative.

Definition 4.1.4 (Interpretation in a Universe of Discourse)

Let P be an atomic n-place predicate symbol. An *interpretation* of P in a universe of discourse $\mathbb{U}$ is an assignment $\mathscr{I}\colon\ P \mapsto \left(P^{\mathbb{U}}, \varGamma_{P^{\mathbb{U}}}\right)$, where $P^{\mathbb{U}}$ is an n-place predicate on $\mathbb{U}$ and $\varGamma$ is a comprehension of $P^{\mathbb{U}}$. Such an

> interpretation is always taken to entail an assignment of $\overline{P}$ to $\left(\overline{P}^{\mathbb{U}}, \mathit{\Gamma}_{\overline{\boldsymbol{P}}^{\mathbb{U}}}\right)$, with $\mathit{\Gamma}_{\boldsymbol{P}^{\mathbb{U}}} = \mathit{\Gamma}_{\overline{\boldsymbol{P}}^{\mathbb{U}}}$.
>
> If $\mathit{\Sigma}$ is a set sentences, then an interpretation of $\mathit{\Sigma}$ in a universe of discourse $\mathbb{U}$ is a set consisting of one interpretation in $\mathbb{U}$ of each atomic predicate symbol P occurring among the sentences in $\mathit{\Sigma}$, such that if two distinct predicate symbols are assigned to the same predicate by their interpretation, then they are also assigned to the same comprehension for that predicate.

In what follows, we will concern ourselves almost exclusively with interpretations in which the comprehensions are at least cumulative. Also, we will sometimes loosely refer to an interpretation "having" a specific knowledge property, e.g., we will have occasion to speak of a "set-theoretic interpretation." In these cases we always mean that the comprehensions that comprise the interpretation all have the mentioned knowledge property.

> Definition 4.1.5 (Choices)
>
> Let $P\tau_0, \ldots, \tau_{n-1}$ be an atomic literal interpreted in a universe of discourse $\mathbb{U}$, and let $\left(P^{\mathbb{U}}, \mathit{\Gamma}_{\boldsymbol{P}^{\mathbb{U}}}\right)$ be the interpretation of P. We define *choices* for the indeterminates occurring among the τ_i by recursion on their dependency relation. Let $\dot{x}$ be the free indeterminate occurring among the τ_i, and let $\beta \subseteq n$ be the set of indices for those argument places of $P\tau_0, \ldots, \tau_{n-1}$ in which $\dot{x}$ occurs. Then an object $b \in \mathbb{U}$ is said to be a *choice* for $\dot{x}$ in $P\tau_0, \ldots, \tau_{n-1}$ if $b \in \gamma_{<\emptyset,\beta,\boldsymbol{P},\mathbb{U}>}(< >)$. ("$< >$" denotes the empty sequence.) If $\dot{x}^A$ is any indeterminate occurring among the τ_i, $\alpha \subseteq n$ is the set of indices for the argument places of $P\tau_0, \ldots, \tau_{n-1}$ in which the indeterminates whose basic symbols are contained in A occur, $\beta \subseteq n \setminus \alpha$

is the set of indices for the argument places in which $\dot{x}^A$ occurs, and $< a_{\alpha_1}, \ldots, a_{\alpha_i} >$ is a tuple of prior choices for the indeterminates on which $\dot{x}^A$ depends, then $b \in \mathbb{U}$ is a *choice* for $\dot{x}^A$ in $P\tau_0, \ldots, \tau_{n-1}$ if $b \in \gamma_{<\alpha,\beta,P,\mathbb{U}>}(< a_{\alpha_1}, \ldots, a_{\alpha_i} >)$.

Definition 4.1.6 (Restricted-Choice Trees for Atomic Literals)

Let $P\tau_0, \ldots, \tau_{n-1}$ be an atomic literal interpreted in a universe of discourse $\mathbb{U}$, and let $\left(P^{\mathbb{U}}, \mathit{\Gamma}_{P^{\mathbb{U}}}\right)$ be the interpretation of P. Suppose $I_P = \{\dot{x}_1^{A_1}, \ldots, \dot{x}_k^{A_k}\}_{1 \leq k \leq n}$ is the set of indeterminates occurring among the τ_i. The choices in $P\tau_0, \ldots, \tau_{n-1}$ available to the indeterminates in I_P form a tree of height $h \leq k + 1$, where level 0 is the root of the tree, level 1 consists of a node for each choice available to the free indeterminate, level 2 consists of sets of nodes, one for each prior choice of the free indeterminate, with nodes in each set corresponding to the choices available to the next indeterminate in the dependency relation given the prior choice of the free indeterminate, and so on. Each branch of the tree of height $k + 1$ (if any) corresponds to a distinct n-tuple $< a_0, \ldots, a_{n-1} >$ of elements of $\mathbb{U}$, where a_i is the corresponding choice of the indeterminate occurring in the i^{th} argument place of $P\tau_0, \ldots, \tau_{n-1}$. We will denote by $\mathcal{T}_{<P,I_P,P^{\mathbb{U}},\mathit{\Gamma}_{P^{\mathbb{U}}}>}$ the subtree consisting of just those branches corresponding to elements of $dom(P^{\mathbb{U}})$. We call $\mathcal{T}_{<P,I_P,P^{\mathbb{U}},\mathit{\Gamma}_{P^{\mathbb{U}}}>}$ the *restricted-choice tree* for $P\tau_0, \ldots, \tau_{n-1}$ under the given interpretation, and we refer to the nodes of this tree as the *restricted choices* for the indeterminates in I_P. It may happen that $\mathcal{T}_{<P,I_P,P^{\mathbb{U}},\mathit{\Gamma}_{P^{\mathbb{U}}}>}$ is empty, i.e., that no complete sequence of choices can be made, either because one of the knowledge functions has an empty image, or because no available sequence of choices lies in the domain of $P^{\mathbb{U}}$. In this case we would say that the comprehension of $P^{\mathbb{U}}$ is *inadequate* for $P\tau_0, \ldots, \tau_{n-1}$, or equivalently that $P\tau_0, \ldots, \tau_{n-1}$ is *undecidable* in $\mathbb{U}$ under the given interpretation. If $\mathcal{T}_{<P, I_P, P^{\mathbb{U}}, \Gamma_{P^{\mathbb{U}}}>}$ is not empty, we will say that $P\tau_0, \ldots, \tau_{n-1}$ is *decidable* in $\mathbb{U}$ under the given interpretation.

Definition 4.1.7 (Strategies for Atomic Literals)

Let $P\tau_0, \ldots, \tau_{n-1}$ be an atomic literal interpreted in a universe of discourse $\mathbb{U}$, let $\left(P^{\mathbb{U}}, \Gamma_{P^{\mathbb{U}}}\right)$ be the interpretation of P, and let $I_P = \{\dot{x}_1^{A_1}, \ldots, \dot{x}_k^{A_k}\}_{1 \leq k \leq n}$ be the set of indeterminates occurring among the τ_i. Suppose that $P\tau_0, \ldots, \tau_{n-1}$ is decidable under the given interpretation. We define a 2-team, k-player game on $\mathcal{T}_{<P, I_P, P^{\mathbb{U}}, \Gamma_{P^{\mathbb{U}}}>}$ as follows. Each indeterminate in I_P is a player. The team called the *Witnesses* consists of the witnesses in I_P, and the team called the *Falsifiers* consists of the falsifiers in I_P. We permit teams to have 0 players. Each player gets 1 turn, with the free indeterminate playing first, and the other indeterminates playing in order of dependence. A turn for a player consists of that player selecting one element of $\mathbb{U}$ from among its restricted choices for $P\tau_0, \ldots, \tau_{n-1}$, based on the previous choices of any indeterminates upon which the player depends. Every game terminates with the selection of an n-tuple $< a_0, \ldots, a_{n-1} >$, where a_i is the corresponding choice of the indeterminate occurring in the i^{th} argument place of $P\tau_0, \ldots, \tau_{n-1}$. The Witnesses win the game if $P^{\mathbb{U}} a_0, \ldots, a_{n-1} = true$, and the Falsifiers win the game if $P^{\mathbb{U}} a_0, \ldots, a_{n-1} = false$.

Since this is a finite game, it is determined, i.e., either the Witnesses or the Falsifiers have a strategy. If the Witnesses have a strategy we will call each choice made by a witness in the execution of that strategy a *strategic choice* for that witness. Similarly, we will call each choice made by a falsifier in the execution of a Falsifier strategy a *strategic choice* for that falsifier.

Definition 4.1.8 (Compatible Strategies)

Suppose that two literals occur in a sentence and that the Witnesses (or Falsifiers) have a strategy for each literal. Then these strategies are said to be *compatible* if any witnesses (resp. falsifiers) that occur among the arguments of both literals make the same strategic choices in both strategies. A collection of more than two strategies is compatible if the strategies it contains are pairwise compatible.

Definition 4.1.9 (Strategies for Sentences)

Let φ be a flat sentence in conjunctive form interpreted in a universe of discourse $\mathbb{U}$, and let I_φ be the set of indeterminates occurring among the literals in φ. If C is a clause in φ then the Witnesses have a strategy for C if and only if the Witnesses have a strategy for at least one literal occurring in C. The Witnesses have a strategy for φ if and only if the Witnesses have a strategy for each clause in φ and the collection of these strategies is compatible.

Dually, if C is a clause in φ then the Falsifiers have a strategy for C if and only if the Falsifiers have a strategy for each literal occurring in C and the collection of these strategies is compatible. The Falsifiers have a strategy for φ if and only if the Falsifiers have a strategy for at least one clause in φ.

> The same definitions hold, *mutatis mutandis*, for flat sentences in disjunctive form. Strategies for sentences with depth greater than 1 are defined by recursion on their depth.
>
> Definition 4.1.10 (Truth in a Universe of Discourse)
>
> Let φ be a sentence interpreted in a universe of discourse $\mathbb{U}$. Then φ is said to be *true in* $\mathbb{U}$ under the interpretation, denoted by $\mathbb{U} \models \varphi$, if and only if the Witnesses have a strategy for φ in $\mathbb{U}$. Also, φ is said to be *false in* $\mathbb{U}$ under the interpretation, denoted by $\mathbb{U} \models \overline{\varphi}$, if and only if the Falsifiers have a strategy for φ in $\mathbb{U}$. If neither the Witnesses nor the Falsifiers have a strategy for φ in $\mathbb{U}$, i.e., if $\mathbb{U} \not\models \varphi$ and $\mathbb{U} \not\models \overline{\varphi}$, then φ is said to be *undecidable* in $\mathbb{U}$ under the given interpretation.

It is clear that no sentence can be both true and false simultaneously, under any interpretation, since witness and falsifier strategies are mutually exclusive. However, in Hologic the law of excluded middle does not hold because literals – and consequently whole sentences – may have inadequate comprehensions. That is, we have

$$\begin{aligned} & \mathbb{U} \models \varphi \Rightarrow \mathbb{U} \not\models \overline{\varphi} \\ \text{and} \quad & \mathbb{U} \models \overline{\varphi} \Rightarrow \mathbb{U} \not\models \varphi, \end{aligned}$$

but the converses are in general false:

$$\begin{aligned} & \mathbb{U} \not\models \varphi \not\Rightarrow \mathbb{U} \models \overline{\varphi} \\ \text{and} \quad & \mathbb{U} \not\models \overline{\varphi} \not\Rightarrow \mathbb{U} \models \varphi. \end{aligned}$$

In light of this, we can preserve a notion of validity for sentences only by restricting the notion to cases where the comprehensions are adequate.

Definition 4.1.11 (Validity)

A sentence φ is said to be *valid*, denoted $\models\varphi$, if $\mathbb{U} \models \varphi$ for every universe of discourse $\mathbb{U}$ and every interpretation into $\mathbb{U}$ under which φ is decidable. Dually, a sentence φ is said to be a *contradiction*, denoted $\models \overline{\varphi}$, if $\mathbb{U} \models \overline{\varphi}$ for every universe of discourse $\mathbb{U}$ and every interpretation into $\mathbb{U}$ under which φ is decidable.

We thus have $\models \varphi \Leftrightarrow \not\models \overline{\varphi}$.

Chapter 5 REASONING WITH HOLOGIC

Our aim in this chapter is to take the first steps towards a proof system for Hologic, and to outline some promising areas of future research.

As before we use capital Roman letters like I and J to denote sets of indeterminates. Also, we may occasionally continue to use lower case Greek letters such as φ and ψ informally as meta-symbols for sentences or formulas. However, we now adopt the convention of formally denoting sentences by combining complex predicate symbols such as $\mathcal{P}$ and $\mathcal{Q}$ — because sentences are (generally complex) literals — with a subscripted letter to represent the set of indeterminates whose elements fill the argument places of $\mathcal{P}$ in some specific manner (i.e., according to a mapping that is assumed and left implicit). Thus, “$\mathcal{P}_I$” denotes a sentence whose complex predicate symbol is $\mathcal{P}$ and whose argument places are filled in some specific manner by the elements of the set of indeterminates I. Additionally, we will use combined forms such as $I_{\mathcal{P}}$ to indicate that I is a set of indeterminates *and* that the members of I fill the argument places of the literals in $\mathcal{P}$ in some specific manner.

These notational conventions allow the possibility that even if two sets of indeterminates I and J are identical, nonetheless $\mathcal{P}_I$ and $\mathcal{P}_J$ may be very different sentences. For instance, we may have $I = J = \{\check{x}, \hat{y}^x\}$ but $\mathcal{P}_I = \mathcal{P}\check{x}\hat{y}^x$ and $\mathcal{P}_J = \mathcal{P}\hat{y}^x\check{x}$. In this case we would write $I_{\mathcal{P}} \neq J_{\mathcal{P}}$.

We prefer the simplicity of this approach to that of introducing new symbols to explicitly represent the mappings that assign a given set of indeterminates to the argument places of a predicate.

Section 5.1 Strategic Implication

In this section we aim to characterize the rules for Hologic that are the counterparts to the standard first-order logic rules ($\mathcal{L}$-rules) for the manipulation of quantifiers, such as $\forall x Px \Rightarrow \exists x Px$ and $(\exists x Px \vee \exists y Qy) \Leftrightarrow \exists x(Px \vee Qx)$. In $\mathcal{L}$ these rules and equivalences rest on the usual model-theoretic interpretations of the quantifiers. In Hologic such rules — where they hold — must be derived from the semantics of strategies.

Definition 5.1.1 (Strategic Implication) If $\mathcal{P}$ is a predicate symbol and if $\mathbb{U} \models \mathcal{P}_J$ whenever $\mathbb{U} \models \mathcal{P}_I$, for every universe of discourse $\mathbb{U}$ and every interpretation $\mathscr{I}\colon\ P \mapsto \left(P^{\mathbb{U}}, \Gamma_{P^{\mathbb{U}}}\right)$ for which both $\mathcal{P}_I$ and $\mathcal{P}_J$ are decidable, then we will say that $\mathcal{P}_I$ *strategically implies* $\mathcal{P}_J$, denoted $\mathcal{P}_I \overset{s}{\to} \mathcal{P}_J$, and that $I_{\mathcal{P}}$ *strategically implies* $J_{\mathcal{P}}$, denoted $I_{\mathcal{P}} \overset{s}{\to} J_{\mathcal{P}}$. If both $\mathcal{P}_I \overset{s}{\to} \mathcal{P}_J$ and $\mathcal{P}_J \overset{s}{\to} \mathcal{P}_I$ then we will say that $\mathcal{P}_I$ and $\mathcal{P}_J$ are *strategically equivalent* and denote this by $\mathcal{P}_I \overset{s}{\leftrightarrow} \mathcal{P}_J$ (and also by $I_{\mathcal{P}} \overset{s}{\leftrightarrow} J_{\mathcal{P}}$).

We make the following further distinctions: If $\mathbb{U} \models \mathcal{P}_J$ whenever $\mathbb{U} \models \mathcal{P}_I$, for every universe of discourse $\mathbb{U}$ and

. . . every cumulative interpretation, then we write $\mathcal{P}_I \xrightarrow{cum} \mathcal{P}_J$, etc.

. . . every theoretic interpretation, then we write $\mathcal{P}_I \xrightarrow{thry} \mathcal{P}_J$, etc.

. . . every set-theoretic interpretation, then we write $\mathcal{P}_I \xrightarrow{set} \mathcal{P}_J$, etc.

. . . every universal interpretation, then we write $\mathcal{P}_I \xrightarrow{uni} \mathcal{P}_J$, etc.

Of course we have

$$(\mathcal{P}_I \overset{s}{\to} \mathcal{P}_J) \Rightarrow (\mathcal{P}_I \xrightarrow{cum} \mathcal{P}_J) \Rightarrow (\mathcal{P}_I \xrightarrow{thry} \mathcal{P}_J) \Rightarrow (\mathcal{P}_I \xrightarrow{set} \mathcal{P}_J) \Rightarrow (\mathcal{P}_I \xrightarrow{uni} \mathcal{P}_J).$$

Example 5.1.2 For every unary predicate $\mathcal{P}$ and every universe $\mathbb{U}$ we have $\mathbb{U} \models \mathcal{P}\hat{x} \Rightarrow \mathbb{U} \models \mathcal{P}\check{x}$. For if the sentence is decidable then there is at least one choice for the single argument place of P, but, given $\mathbb{U} \models \mathcal{P}\hat{x}$, there is evidently not a strategic choice for the Falsifiers, so any choice must be a strategic choice for the Witnesses. So $\mathbb{U} \models \mathcal{P}\check{x}$. Consequently $\mathcal{P}\hat{x} \overset{s}{\to} \mathcal{P}\check{x}$.

There are no free variables in Hologic, so the $\mathcal{L}$-rules for the introduction of quantifiers to bind free variables, such as universal generalization, are not relevant. Also, both quantification and negation in Hologic are notationally infix, so the rules for commuting quantifiers with negation have been adequately dealt with by Definition 3.3.5. In this section we will examine the counterparts in Hologic to the following standard $\mathcal{L}$-rules and equivalences affecting quantifiers:

1. $\forall x\, \varphi x \Rightarrow \exists x\, \varphi x$,
2. $\exists x \forall y\, \varphi x y \Rightarrow \forall y \exists x\, \varphi x y$,
3. $\exists x \exists y\, \varphi x y \Leftrightarrow \exists y \exists x\, \varphi x y$,
4. $\forall x \forall y\, \varphi x y \Leftrightarrow \forall y \forall x\, \varphi x y$,
5. $\exists x(\varphi x \vee \psi x) \Leftrightarrow (\exists x\, \varphi x \vee \exists y\, \psi y)$, and
6. $\forall x(\varphi x \wedge \psi x) \Leftrightarrow (\forall x\, \varphi x \wedge \forall y\, \psi y)$.

We take them in order, and we will show in addition that (2) may fail if the interpretation is not cumulative, and (3) and (4) may fail if the interpretation is not theoretic.

Lemma 5.1.3 (Flipping Down)

Let $\mathcal{P}_I$ be a sentence and suppose $\hat{x}^{\mathcal{A}}$ is a falsifier occurring in $I_{\mathcal{P}}$. Let $J_{\mathcal{P}}$ be obtained from $I_{\mathcal{P}}$ by replacing $\hat{x}^{\mathcal{A}}$ with $\check{x}^{\mathcal{A}}$. (This is called, informally, "flipping down.") Then $\mathcal{P}_I \overset{s}{\to} \mathcal{P}_J$.

> *Proof:* If $\mathbb{U} \models \mathcal{P}_I$ then no choice by $\hat{x}^A$ is a strategic choice for the Falsifiers, so any choice by $\check{x}^A$ is a strategic choice for the Witnesses. If $\hat{x}^A$ has no choice, but still $\mathbb{U} \models \mathcal{P}_I$, then evidently the Witnesses retain the same strategy for $\mathcal{P}_J$ if $\check{x}^A$ has no choice.

Definition 5.1.4 (Adjacent Indeterminates)

If $\dot{x}^A$ and $\dot{y}^B$ are indeterminates occuring in a set of indeterminates $I_\mathcal{P}$ with $\dot{x}^A \prec \dot{y}^B$, then we will say that $\dot{x}^A$ and $\dot{y}^B$ are *adjacent* if there is no $\dot{z}^C \in I_\mathcal{P}$ such that $\dot{x}^A \prec \dot{z}^C \prec \dot{y}^B$.

Lemma 5.1.5 (Changing Down)

Let $\mathcal{P}_I$ be a sentence and suppose $\check{x}^A$ and $\hat{y}^B$ are adjacent indeterminates occurring in $I_\mathcal{P}$ with $\hat{y}^B \prec \check{x}^A$. Let $J_\mathcal{P}$ be obtained from $I_\mathcal{P}$ by deleting the basic symbol x from the argument B of $\hat{y}^B$ and adding the basic symbol y to the argument A of $\check{x}^A$, so that $\check{x}^A \prec \hat{y}^B$. (This is called, informally, “changing down.”) Then $\mathcal{P}_I \xrightarrow{cum} \mathcal{P}_J$.

> *Proof:* Suppose $\mathbb{U} \models \mathcal{P}_I$. If $\check{x}^A$ has a choice for $\mathcal{P}_I$ then it has a strategic choice for $\mathcal{P}_I$, i.e., $\check{x}^A$ can make a choice such that no subsequent choice by $\hat{y}^B$ results in a win for the Falsifiers. Since the interpretation is cumulative, $\check{x}^A$ can make the same choice in $\mathcal{P}_J$, but $\hat{y}^B$ has no choices available to it in $\mathcal{P}_J$ that were not available to it in $\mathcal{P}_I$. Consequently the same choice by $\check{x}^A$ is strategic for $\mathcal{P}_J$. If on the other hand $\check{x}^A$ can make no choice in $\mathcal{P}_I$ then evidently the Witness strategy does not depend on an evaluation of any literal in which $\check{x}^A$ occurs, and the commutation of $\check{x}^A$ and $\hat{y}^B$ does not change the choices available to $\hat{y}^B$

in any literal in which $\check{x}^{A}$ does not occur. Hence the Witnesses retain the same strategy for $\mathcal{P}_{J}$.

Changing down can fail under a non-cumulative interpretation. For example, let $\mathbb{U} = \{a, b\}$ and let $P^{\mathbb{U}}$ be (given by the characteristic function on) the binary relation $\{(a, a), (a, b)\}$ on $\mathbb{U}$. We interpret the binary predicate symbol P by $\left(P^{\mathbb{U}}, \Gamma_{P^{\mathbb{U}}}\right)$, where $\Gamma_{P^{\mathbb{U}}}$ is the following set of knowledge functions. (An asterisk marks the argument place(s) for which a choice from the image set is to be made.)

$$\begin{aligned}
\gamma_{<\emptyset,\{0\},P^{\mathbb{U}},\Gamma_{P^{\mathbb{U}}}>} &: \; < \underline{\;*\;} , \underline{\quad} > \;\mapsto\; \{a, b\}. \\
\gamma_{<\emptyset,\{1\},P^{\mathbb{U}},\Gamma_{P^{\mathbb{U}}}>} &: \; < \underline{\quad} , \underline{\;*\;} > \;\mapsto\; \{a, b\}. \\
\gamma_{<\emptyset,\{0,1\},P^{\mathbb{U}},\Gamma_{P^{\mathbb{U}}}>} &: \; < \underline{\;*\;} , \underline{\;*\;} > \;\mapsto\; \{a, b\}. \\
\gamma_{<\{0\},\{1\},P^{\mathbb{U}},\Gamma_{P^{\mathbb{U}}}>} &: \; < \underline{\;a\;} , \underline{\;*\;} > \;\mapsto\; \{a, b\}, \\
&\; < \underline{\;b\;} , \underline{\;*\;} > \;\mapsto\; \{a, b\}. \\
\gamma_{<\{1\},\{0\},P^{\mathbb{U}},\Gamma_{P^{\mathbb{U}}}>} &: \; < \underline{\;*\;} , \underline{\;a\;} > \;\mapsto\; \{a, b\}, \\
&\; < \underline{\;*\;} , \underline{\;b\;} > \;\mapsto\; \{b\}.
\end{aligned}$$

Then $P\check{x}\hat{y}^{x}$ is true but $P\check{x}^{y}\hat{y}$ is false. This is because the witness $\check{x}$ plays first in $P\check{x}\hat{y}^{x}$ and may pick a for the first argument place of P — since a is contained in the image of $\gamma_{<\emptyset,\{0\},P^{\mathbb{U}},\Gamma_{P^{\mathbb{U}}}>}(< \underline{\;*\;} , \underline{\quad} >)$ — so $\hat{y}^{x}$ is subsequently forced to pick one of a or b, and for either choice the result (Paa or Pab) is true. However, the falsifier $\hat{y}$ plays first in $P\check{x}^{y}\hat{y}$ and may pick b for the second argument place of P, and this forces $\check{x}^{y}$ to subsequently pick b for the first argument place as well, since b is the only element in the image of $\gamma_{<\{1\},\{0\},P^{\mathbb{U}},\Gamma_{P^{\mathbb{U}}}>}(< \underline{\;*\;} , \underline{\;b\;} >)$. The result ($Pbb$) is false.

We turn next to the commutation of like quantifiers.

Lemma 5.1.6 (Commutating Like Indeterminates)

Let $\mathcal{P}_{\boldsymbol{I}}$ be a sentence and suppose $\dot{x}^{\boldsymbol{A}}$ and $\dot{y}^{\boldsymbol{B}}$ are adjacent witnesses or adjacent falsifiers occurring in I, with $\dot{y}^{\boldsymbol{B}} \prec \dot{x}^{\boldsymbol{A}}$. Let $J_{\mathcal{P}}$ be obtained from $I_{\mathcal{P}}$ by deleting the basic symbol x from the argument B of $\dot{y}^{\boldsymbol{B}}$ and adding the basic symbol y to the argument A of $\dot{x}^{\boldsymbol{A}}$, so that $\dot{x}^{\boldsymbol{A}} \prec \dot{y}^{\boldsymbol{B}}$. (This is called *commutating* $\dot{x}^{\boldsymbol{A}}$ and $\dot{y}^{\boldsymbol{B}}$.) Then $\mathcal{P}_{\boldsymbol{I}} \overset{thry}{\longleftrightarrow} \mathcal{P}_{\boldsymbol{J}}$.

> *Proof:* In any literal in $\mathcal{P}_{\boldsymbol{I}}$ in which both indeterminates occur the interpretation of the predicate symbol is assumed to be theoretic, and consequently the choices for the argument places occupied by each indeterminate are the same before and after the argument places occupied by the other indeterminate are filled. (*A fortiori* this is true in predicates in which only one of the indeterminates occurs, or in which neither occurs.) Consequently, the order in which adjacent, like indeterminates play can have no effect on any Witness or Falsifier strategy.

Commutation of like quantifiers can fail for non-theoretic interpretations: Let $\mathbb{U} = \{a, b\}$ and let $P^{\mathbb{U}}$ be (given by the characteristic function on) the binary relation $\{(a, b), (b, b)\}$ on $\mathbb{U}$. We interpret the binary predicate symbol P by $\left(P^{\mathbb{U}}, \mathit{\Gamma}_{\boldsymbol{P}^{\mathbb{U}}}\right)$, where $\mathit{\Gamma}_{\boldsymbol{P}^{\mathbb{U}}}$ is the following set of knowledge functions.

$$\begin{aligned}
\gamma_{<\emptyset,\{0\},P^{\mathbb{U}},\Gamma_{P^{\mathbb{U}}}>} &: < \underline{*}\,, \underline{} > \;\mapsto\; \{a,b\}.\\
\gamma_{<\emptyset,\{1\},P^{\mathbb{U}},\Gamma_{P^{\mathbb{U}}}>} &: < \underline{}\,, \underline{*} > \;\mapsto\; \{a\}.\\
\gamma_{<\emptyset,\{0,1\},P^{\mathbb{U}},\Gamma_{P^{\mathbb{U}}}>} &: < \underline{*}\,, \underline{*} > \;\mapsto\; \{a,b\}.\\
\gamma_{<\{0\},\{1\},P^{\mathbb{U}},\Gamma_{P^{\mathbb{U}}}>} &: < \underline{a}\,, \underline{*} > \;\mapsto\; \{a,b\},\\
& \; < \underline{b}\,, \underline{*} > \;\mapsto\; \{a,b\}.\\
\gamma_{<\{1\},\{0\},P^{\mathbb{U}},\Gamma_{P^{\mathbb{U}}}>} &: < \underline{*}\,, \underline{a} > \;\mapsto\; \{a,b\},\\
& \; < \underline{*}\,, \underline{b} > \;\mapsto\; \{a,b\}.
\end{aligned}$$

This comprehension is cumulative but not theoretic, and we note that under this interpretation $P\check{x}\,\check{y}^{x}$ is true, but $P\check{x}^{y}\,\check{y}$ is false. For if $\check{x}$ plays first it may choose either a or b and $\check{y}$ may subsequently pick b, so the Witnesses have a strategy for $P\check{x}\,\check{y}^{x}$. However, if $\check{y}$ must play first then it's only choice is a, leaving no alternative for $\check{x}$ but to make a choice that falsifies P, so the Falsifiers have a strategy for $P\check{x}^{y}\,\check{y}$.

Finally we examine the Hologic counterparts to the equivalences $\exists x(\varphi x \vee \psi x) \Leftrightarrow (\exists x\, \varphi x \vee \exists y\, \psi y)$, and$\forall x(\varphi x \wedge \psi x) \Leftrightarrow (\forall x\, \varphi x \wedge \forall y\, \psi y)$.

Lemma 5.1.7 (Unifying Witnesses)

Suppose that in a sentence $\mathcal{P}_I$ there occurs a disjunction $\mathcal{Q}_K \vee \mathcal{R}_L$ such that the witness $\check{x}_1^{A_1}$ is local and free for $K_{\mathcal{Q}}$ and the witness $\check{x}_2^{A_2}$ is local and free for $L_{\mathcal{R}}$. Let $J_{\mathcal{P}}$ be obtained from $I_{\mathcal{P}}$ by choosing a new basic symbol x_3 and replacing each occurrence of $\check{x}_1^{A_1}$ and each occurrence of $\check{x}_2^{A_2}$ in $I_{\mathcal{P}}$ with the indeterminate $\check{x}_3^{A_3}$, where $A_3 = A_1 \cup A_2$, and replacing each non-root occurrence among the indeterminates in $I_{\mathcal{P}}$ of either of the basic symbols x_1 and x_2 with the new basic symbol x_3. Then $\mathcal{P}_I \overset{s}{\leftrightarrow} \mathcal{P}_J$.

Proof: We observe that since $\check{x}_1^{A_1}$ is local and free for $K_{\mathcal{Q}}$ and $\check{x}_2^{A_2}$ is local and free for $L_{\mathcal{R}}$ then neither of $\check{x}_1^{A_1}$ and $\check{x}_2^{A_2}$ can be dependent on the other. Consequently, if $I_{\mathcal{P}}$ is a rooted and well-founded set of indeterminates, then $J_{\mathcal{P}}$ is also rooted and well-founded, and so $\mathcal{P}_J$ is well-formed.

Now suppose $\mathbb{U} \models \mathcal{P}_I$. For the Witnesses to have a strategy it is sufficient that just one of $\check{x}_1^{A_1}$ and $\check{x}_2^{A_2}$ have a strategic choice, and if either of them does then $\check{x}_3^{A_3}$ may make the same choice as a strategic choice for the corresponding literal(s) in $\mathcal{P}_J$. On the other hand, if neither $\check{x}_1^{A_1}$ nor $\check{x}_2^{A_2}$ can make a choice in $\mathcal{P}_I$, but nonetheless $\mathbb{U} \models \mathcal{P}_I$, then evidently the Witnesses' strategy did not require a choice by either of these indeterminates, in which case the same Witness strategy will not require a choice by $\check{x}_3^{A_3}$ for $\mathcal{P}_J$.

We observe that when we unify the local, free witnesses in a disjunction, we may take the disjunction to be a new complex literal, for which the newly minted witness is local and may (and may not) be free. Consequently, if there are more than two literals in a clause each of which has a free local witness, we may iterate the unification until all of the qualifying disjuncts are disjoined in a new complex literal for which the witness unifying all is a local trifle, but inside of which it is a non-local trifle for each disjunct. (Recall that a non-local trifle is an indeterminate upon which no other non-local indeterminate depends.)

Of course, what can be unified without cost can be split without cost.

Lemma 5.1.8 (Splitting Witnesses)

Suppose a complex literal $\mathcal{Q}_K$ occuring in a sentence $\mathcal{P}_I$ has the form of a clause (a disjunction of literals), that a witness $\check{x}^A$ is the local trifle for $\mathcal{Q}_K$, and that in each of its disjuncts $\check{x}^A$ is the non-local trifle. Let $\mathcal{P}_J$

be obtained from $\mathcal{P}_I$ by choosing as many new basic symbols $x_1, x_2, \ldots$ as there are disjuncts in $\mathcal{Q}_K$, and eliminating every occurrence of $\check{x}^A$ by replacing its occurrence in the i^{th} disjunct with $\check{x}_i^{A_i}$, where A_i contains just those basic symbols in A that also have root occurrences in the i^{th} disjunct. Then $\mathcal{P}_I \overset{s}{\leftrightarrow} \mathcal{P}_J$.

Proof: If $\mathbb{U} \models \mathcal{P}_I$ and if $\check{x}^A$ makes a strategic choice as part of the Witnesses' strategy for $\mathcal{P}_I$, then, for some i, $\check{x}^A$'s choice is a strategic choice for the i^{th} disjunct in $\mathcal{Q}_K$. In this case $\check{x}_i^{A_i}$ may make the same strategic choice in the i^{th} disjunct as part of the same Witness strategy for $\mathcal{P}_J$. If on the other hand $\check{x}^A$ has no choice in $\mathcal{P}_I$ then evidently the Witness strategy does not depend on the evaluation of any literal in which $\check{x}^A$ occurs, and so the same Witness strategy may be applied in $\mathcal{P}_J$ without regard to the evaluation of any literal in which any of the $\check{x}_i^{A_i}$ occur.

The counterparts in Hologic to the $\mathcal{L}$ equivalence $\forall x(Px \wedge Qx) \equiv (\forall x Px \wedge \forall y Qy)$ are the duals to the above lemmas:

Lemma 5.1.9 (Unifying Falsifiers)

Suppose that in a sentence $\mathcal{P}_I$ there occurs a conjunction $\mathcal{Q}_K \wedge \mathcal{R}_L$ such that the falsifier $\hat{x}_1^{A_1}$ is local and free for $K_\mathcal{Q}$ and the falsifier $\hat{x}_2^{A_2}$ is local and free for $L_\mathcal{R}$. Let $J_\mathcal{P}$ be obtained from $I_\mathcal{P}$ by choosing a new basic symbol x_3 and replacing each occurrence of $\hat{x}_1^{A_1}$ and each occurrence of $\hat{x}_2^{A_2}$ in $I_\mathcal{P}$ with the indeterminate $\hat{x}_3^{A_3}$, where $A_3 = A_1 \cup A_2$, and replacing each non-root occurrence among the indeterminates in $I_\mathcal{P}$ of either of the basic symbols x_1 and x_2 with the new basic symbol x_3. Then $\mathcal{P}_I \overset{s}{\leftrightarrow} \mathcal{P}_J$.

Lemma 5.1.10 (Splitting Falsifiers)

> Suppose a complex literal $\mathcal{Q}_K$ occuring in a sentence $\mathcal{P}_I$ has the form of a conjunction of literals, that a falsifier $\hat{x}^A$ is the local trifle for $\mathcal{Q}_K$, and that in each of its conjuncts $\hat{x}^A$ is the non-local trifle. Let $\mathcal{P}_J$ be obtained from $\mathcal{P}_I$ by choosing as many new basic symbols $x_1, x_2, \ldots$ as there are conjuncts in $\mathcal{Q}_K$, and eliminating every occurrence of $\check{x}^A$ by replacing its occurrence in the i^{th} conjunct with $\hat{x}_i^{A_i}$, where A_i contains just those basic symbols in A that also have root occurrences in the i^{th} conjunct. Then $\mathcal{P}_I \overset{s}{\leftrightarrow} \mathcal{P}_J$.

This completes our current treatment of strategic implication. In view of our remark immediately following Definition 5.1.2, it is clear that many $\mathcal{L}$-rules for the manipulation of quantifiers have no counterpart in Hologic. For instance, there is no means in Hologic of introducing quantification of a variable that nowhere appears as an argument to a predicate. On the other hand, since any rule of strategic implication would have to hold for set-theoretic interpretations, Hologic can have no rules of this kind that do not have counterparts for the manipulation of quantifiers in $\mathcal{L}$.

Section 5.2 Remarks Towards Future Research

In closing, we present several informal remarks on topics in Hologic that seem to us to warrant continuing investigation.

Remark 5.2.1 (The Proof System)

> A proof system for Hologic would use the symbols we have previously defined as part of the formal language. In addition, we would expect to treat the complex predicate symbols and the strategic implication symbols as formal symbols in our deductive calculus. Finally, it may be advanta-

geous to introduce the special variable-arity predicate symbols $\top$ and $\bot$, which would be taken to correspond to the total and the empty relations, respectively, in any interpretation. The complete set of symbols would then consist of:

a. The logical connectives $\vee$ and $\wedge$ and the overline symbol to represent the negation or complementation operation.

b. The strategic implication connective $\overset{s}{\rightarrow}$.

c. The sets of atomic and complex predicate symbols $\{P, Q, R, \ldots\ \overline{P}, \overline{Q}, \overline{R}, \ldots\}$ and $\{\mathcal{P}, \mathcal{Q}, \mathcal{R}, \ldots, \overline{\mathcal{P}}, \overline{\mathcal{Q}}, \overline{\mathcal{R}}, \ldots\}$ together with the special variable-arity predicate symbols $\top$ and $\bot$.

d. The sets of indeterminate symbols $\{\check{x}, \check{y}, \check{z}, \ldots\}$ and $\{\hat{x}, \hat{y}, \hat{z}, \ldots\}$

It seems likely that the following rules of proof will suffice:

Axioms from Tautologies: If $\mathcal{A}(A_1, \ldots, A_n)$ is a propositional tautology and $\mathcal{P}_1, \ldots, \mathcal{P}_n$ are literals with a common set of non-local indeterminates, then $\mathcal{A}(\mathcal{P}_1, \ldots, \mathcal{P}_n)$ is an axiom.

Modus Ponens: $$\frac{\mathcal{P}_I, \overline{\mathcal{P}_I} \vee \mathcal{Q}_J}{\mathcal{Q}_J}$$

Strategic Implication: $$\frac{\mathcal{P}_I, \mathcal{P}_I \overset{s}{\rightarrow} \mathcal{P}_J}{\mathcal{P}_J}$$

Both the soundness and completeness of the proof system will be subject to novel considerations, owing to the flexibility of the semantics and the possibility of an incompletely specified domain. We anticipate, however, that Hologic will be found to be complete. This is owing to the fact that if a sentence is valid then it must be set-theoretically valid, and consequently the completeness of $\mathcal{L}$ itself comes to bear.

Remark 5.2.2 (Inter-interpretability of Hologic and $\mathcal{L}$)

We believe that there is a straightforward procedure for writing any sentence of Hologic as a sentence of $\mathcal{L}$, and *vice versa*, such that when the predicate symbols are interpreted in the same way in the same set-theoretic domains, the resulting sentences have the same truth conditions. As noted above, this will likely be relevant to the completeness of Hologic. It will also make explicit and precise the assertion that the semantics of Hologic is a generalization of the semantics of $\mathcal{L}$.

Remark 5.2.3 (Sorts)

The distinction between set-theoretic interpretations and universal interpretations has not been found to have any impact on strategic implication. We can interpret this fact to mean that the rules of $\mathcal{L}$ do not distinguish domains consisting of a single sort of object from domains consisting of two or more sorts of objects. In contrast, Hologic evidently handles sorts in a very natural way, by means of the knowledge functions. That is, the sort of an object is set by the comprehension of the predicates for which that object may be used, and consequently no additional rules of logic are required to accommodate a theory with more than one sort. This fact, combined with the rule of sentence construction implicit in Definition 3.3.3, makes defining and using new sorts (including sorts with arities) independent of externally supplied sort definitions. The result for knowledge systems built using Hologic as the deductive engine would be the capacity to draw conclusions that implicitly use and explicitly denote previously undefined categories. We believe this may have implications for artificial intelligence applications.

Remark 5.2.4 (Quantificational Independence)

Jaakko Hintikka proposed in 1996 [4] a game-theoretic semantics for first-order logic, with many of the same aims in view as are achieved by Hologic[3]. Among Hintikka's aims was to supply an unambiguous semantical sense to expressions in which there is no specified order to the quantifiers. (Hintikka calls this “independence-friendly" or IF logic.) We observe that quantificational independence may be handled very naturally in Hologic, both syntactically and semantically. It is syntactically natural because the rules of sentence construction may be altered to permit the dependency relation among the indeterminates in a sentence to be a partial order rather than a well-order. It is semantically natural because when two or more indeterminates are not ordered by dependency, they simply make their restricted choices simultaneously, that is, on the basis of the same knowledge functions each would be governed by if the other indeterminate(s) were making *their* choices subsequently. However, quantificational independence in Hologic results in different truth conditions than in Hintikka's game theoretic semantics.

We believe the development and investigation of an independence-friendly generalization of Hologic is warranted, and not only for reasons of theoretical interest. A continuing challenge for artificial intelligence sys-

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[3] This author is indebted to Hintikka for exposing him to the notion of a game theoretic approach to semantics.

tems is coping formally with incomplete information, and forming testable conjectures on the basis of such information. One reduction of this problem would be to see it as the unstructured accumulation of literals and/or presentences, which must then be organized into a semantically coherent set of sentences. Handling quantificational independence would be a central issue.

Remark 5.2.5 (Computational Treatment)

As we noted in the introduction to Part II of this thesis, we have been motivated in part by a perception that Hologic is very amenable to computational treatment. In particular, it is natural to view sentences in Hologic as data structures in a list-processing (lisp) language. It remains one of our principal goals to implement Hologic as a logic engine for knowledge systems.

This completes our introduction of Hologic.

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