A Brief Exposition Of The Catenary Curve

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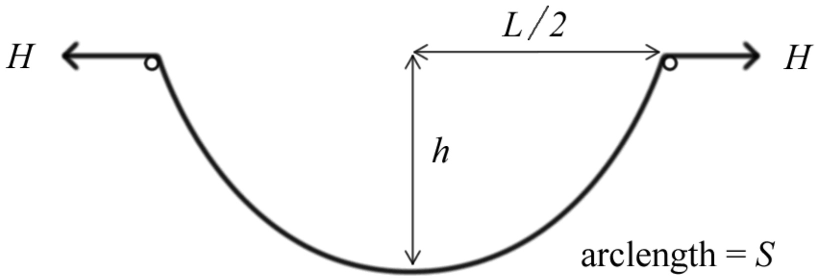
The catenary[1] curve is obtained when a uniform flexible cord or chain hangs by fixed end-points under its own weight. We derive the equations describing the curve by a straightforward application of vector arithmetic and integral calculus, using only the presumption that the forces acting on the cord are in static equilibrium.

It will be seen that for given end-points and a fixed length S of cord the shape of the curve formed by the cord is independent of its weight (provided that it is not zero). However, in practical terms, for given end-points, S itself is determined by the weight w per unit length of the cord and the strength H of the horizontal component of tension in the cord. In particular, the ratio w/H will concern us closely.



`Figure 1`

The amount of “sag” in the curve, represented in Figure 1 by h, can be shown to be approximately $wL^2/8H$, where L is the distance between the end-points. Consequently, there must be at least *some* perceptible sag unless w and L are very small and H is very large, as in the case, for example, of guitar or piano strings.

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[1] The word *catenary* comes from the Latin *catena*, which means “chain.” The term was coined by Christiaan Huygens in 1690.

Although our figure shows the end-points at the same height, this is not necessary. Indeed we could place the end-points anywhere on a given catenary curve, snip the excess, and the resulting portion of the curve would hang unchanged. Put differently, every part of a catenary curve is also a catenary curve.

For simplicity's sake, we place the lowest point of the curve at the origin, with fixed end-points at $x = \pm L/2$ and $y = h$, and consider an incremental piece $\mathtt{d}s$ of the curve. Since the weight of the cord only acts downward, the horizontal component H of the tension in the cord is the same at every point, in both directions. The vertical component V is then given by $V = Hy'$, so $\mathtt{d}V = Hy''$.

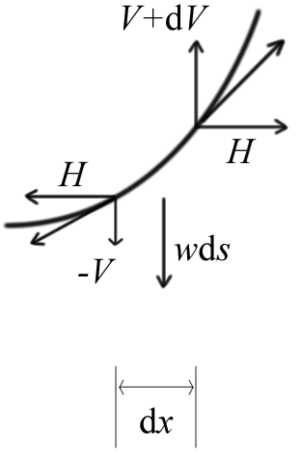

Figure 2

From Figure 2 we deduce that $V + \mathtt{d}V - V - w\,\mathtt{d}s = 0$, that is, $\mathtt{d}V = w\,\mathtt{d}s$. Substituting Hy'' for $\mathtt{d}V$ and $w\sqrt{1+y'^2}\,\mathtt{d}x$ for $w\,\mathtt{d}s$, we get a separable differential equation that is first-order in y'.

$$\begin{aligned} Hy'' &= w\sqrt{1+y'^2}\,\mathtt{d}x \\ \Rightarrow \int \frac{\mathtt{d}y'}{\sqrt{1+y'^2}} &= \int \frac{w}{H}\mathtt{d}x \end{aligned}$$

$$
\begin{aligned}
\Rightarrow \sinh^{-1} y' &= \frac{w}{H}x \\
\Rightarrow y' &= \sinh\left(\frac{w}{H}x\right) \\
\Rightarrow y &= \frac{H}{w}\left(\cosh\frac{w}{H}x - 1\right)
\end{aligned}
$$

In this derivation we have used the fact that both y and y' are zero when x is zero to determine the constants of integration. Typically the ratio $\frac{w}{H}$ is represented by a constant a, so the common form of the equation of the catenary is then $y = \left(\cosh(ax) - 1\right)/a$. Thus we see that the catenary is a hyperbolic cosine curve whose slope is given by the hyperbolic sine, a very elegant finding.[2]

It is convenient to have a form of the equation that uses only the sag distance h and either the span distance L or arc length S, particularly in cases where the horizontal tension and per-unit weight of the cord are not known or are immaterial. Since the arc length is given by $\int \mathtt{d}s$, we have

$$
\begin{aligned}
S &= \int_{-L/2}^{L/2} \mathtt{d}s \\
&= \int_{-L/2}^{L/2} \sqrt{1 + y'^2}\, \mathtt{d}x \\
&= \int_{-L/2}^{L/2} \sqrt{1 + \sinh^2\left(\frac{w}{H}x\right)}\, \mathtt{d}x \\
&= \int_{-L/2}^{L/2} cosh\left(\frac{w}{H}x\right)\, \mathtt{d}x \\
&= 2\frac{H}{w}\, sinh\left(\frac{wL}{2H}\right)
\end{aligned}
$$

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[2] The equation of the catenary is often given as $y = \cosh(ax)/a$, or even just $y = \cosh x$. The difference is that our form of the equation places the vertex at the origin, whereas the others place the "asymptote" of the curve on the x-axis — see the closing remarks.

Also, since $y = \frac{H}{w}\left(\cosh \frac{w}{H}x - 1\right)$ and we placed the lowest point of the curve at the origin, we have

$$h = \frac{H}{w}\left(\cosh\left(\frac{w}{H}\frac{L}{2}\right) - 1\right).$$

Using these two equations and applying the Pythagorean identity, it can easily be shown that

$$a = \frac{w}{H} = \frac{8h}{S^2 - 4h^2},$$

which gives us an explicit representation of a in terms of S and h alone. Consequently, we can determine the curve solely from the arc length and depth of sag, and we can get the corresponding span by using the equation that gave us the arc length and solving it for L:

$$S = \frac{2}{a}\sinh\left(\frac{L}{2}a\right) \;\Rightarrow\; L = \frac{2}{a}\sinh^{-1}\left(\frac{S}{2}a\right)$$

Unfortunately, we cannot solve the equation $h = \left(\cosh(L/2)a - 1\right)/a$ for a, so we are unable to write a as an explicit expression in terms of L and h alone. However, we can put a value for L into the equation for h and then approximate a arbitrarily closely using numerical methods or, more simply, a good graphing calculator.

It should be noted that the curve formed by a suspension cable, such as on bridges or other architectural structures, is actually a parabola — this difference arises from the non-uniform loading placed on the cable. Indeed Galileo had believed that the catenary is a parabola, but this was disproved in 1669. Christiaan Huygens, Gottfried Leibniz, and Johann Bernoulli, all responding to a challenge published by Jacob Bernoulli, derived the correct equations in 1691.

Other notable facts about the catenary include:

1. A catenary is traced by the focus of a parabola rolling along a straight line.
2. It is the locus of the mid-point of the vertical line segment between the curves e^{ax} and e^{-ax}.

3. If we take the catenary as given by the average of the curves e^{ax} and e^{-ax} in the manner just mentioned, then the equation of the catenary is $y = (\cosh(ax))/a$, and the horizontal asymptote of e^{ax} (the x-axis) is called the asymptote of the catenary. The surface obtained by rotating a catenary about its asymptote is called a *catenoid*, and this was proved by Euler in 1744 to be the only minimal surface of revolution.
4. The Gateway Arch in St. Louis, Missouri is an inverted catenary. It is 630 feet wide at the base and 630 feet tall, and its equation is $y = 68.8(\cosh 0.01x - 1)$.
5. The involute of a catenary is a tractrix.
6. The centroid of a regular polygon that rolls on a "road" made up of truncated catenary arches of the correct size will trace a straight line.

I am indebted to J. B. Calvert, whose exposition is published on the world wide web at http://www.du.edu/ jcalvert/math/catenary.htm, for the use of his excellent notation and lucid derivations.