Hypersets

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To the Chairman of Examiners for Part III Mathematics.

Dear Sir,

I enclose the Part III essay of my Tutorial pupil Becker Sidney Smith.

H. Mason, Mathematics Tutor, St. Edmund's.

1 Introduction

Replacing the axiom of foundation in classical set theory with a more flexible alternative is not a new idea. The notion of non-well-founded sets (which in this paper, at the risk of being trendy, I will generically refer to as *hypersets*) had currency among set theorists and logicians from at least the first quarter of the 20th century. Miramanoff, in particular, formulated the fundamental distinction between well-founded sets and hypersets in 1917 [16]. In the 1950's the independence of the axiom of foundation was established, for example by Bernays [4], and since that time many axiomatizations of set theory without a foundation axiom have been put forward [1]. However, motivated largely by antimonies such as Russell's paradox, and by the successes of the cumulative heirarchy of Von Neumann, of the axiom system ZFC, and of the other theories of well-founded sets in justifying the constructions of work-a-day mathematics, most set theorists and logicians came to view hypersets as, at best, a sideshow [3].

This situation has changed. Computer science, linguistics, situation semantics, and the broad fields of artificial intelligence and cognitive science have given urgency to the need to formulate models of self-referential structures. To many, hypersets are the natural avenue to pursue. Some have gone so far as to claim for the theory of hypersets the status of a revolution, e.g., that it extends our understanding of set theory in ways analogous to how the discovery of irrationals by the ancient Greeks, or of complex numbers in the golden age of analysis, extended our understanding of the concept of number [3]. Others, by way of contrast, verge on the dismissive. Thomas Forster, in *Set Theory with a Universal Set*, writes,

> After all, showing that an arbitrary plausible graph can be turned into the $\in$-diagram of a set not otherwise of any set-theoretical interest is not — obviously — of any set-theoretical interest [8].

This paper will not address such larger questions. My purpose here is exposition: to present the theory of hypersets as I have found it in the literature, relying heavily but not exclusively on the work of Peter Aczel [1], to demonstrate its equiconsistency with ZFC, to provide a model for it, and to sketch how it is applied. In doing so, I have departed from the order of presentation used in the primary sources, delaying the discussion of specific anti-foundation axioms (such as Aczel's AFA) until after a formal development of the machinery needed to prove the central results.

2 Motivating Hypersets

A simple and intuitively appealing motivation for hypersets arises out of the standard notion of limits. For instance, students of analysis are at some point required to evaluate continued fractions, such as the following:

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \dots}}}$$

Formally, this must be approached by taking a limit, as, for example, of the sequence

$$x_1 = 1 + \frac{1}{1+1},\ x_2 = 1 + \frac{1}{1 + \frac{1}{1+1}},\ x_3 = 1 + \frac{1}{1 + \frac{1}{1+\frac{1}{1+1}}},\ \dots$$

With lots of perserverence, one can prove that $\lim_{n \longrightarrow \infty} x_n = \frac{1+\sqrt{5}}{2}$, i.e., the golden ratio. However, there is a simpler approach, which is to observe the identity

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \dots}}} \iff x = 1 + \frac{1}{x}$$

and then simply solve the resulting quadratic equation. Intuitively, this boils down to seeing that any denominator on the right hand side of the original equation is actually *equal to the entire expression* on the right hand side of the original equation.

A thoroughly analogous situation arises naturally in sets. Consider the sequence of sets,

$$\begin{aligned}
x_1 &= \{0\} \\
x_2 &= \{0, \{0\}\} \\
x_3 &= \{0, \{0, \{0\}\}\} \\
x_4 &= \{0, \{0, \{0, \{0\}\}\}\} \\
&\vdots \\
x_n &= \{0, \{0, \{0, \{0, \dots \{0\} \dots\}\}\}\} \\
&\vdots
\end{aligned}$$

Does it make any sense to talk about a limit of this sequence? In other words, can we discuss the result of iterating this construction ω (countably infinitely many) times? Although each set in the sequence is well-founded, it is clear that the limit set

$$x = \lim_{n \longrightarrow \infty} x_n = \{0, x\}$$

is not well founded, because it has an infinite descending $\in$-chain, and in particular has itself as a member. Finding a way to talk sense about such a set is what hyperset theory is about.

Definition 2.1
A *hyperset* is a set that is not well-founded, i.e., that has an infinite descending $\in$-chain.

A construction like the one above is not whimsical or impractical; it arises naturally in computer science, for instance, in the structure known as a *stream*. This is a recursive data-type of the form $s = \langle a, s^* \rangle$, where s^* is another stream.

Another intuitively appealing (and currently fashionable) motivation for hypersets arises from self-referential phenomena, which are often modeled by non-well-founded relations. Consider, for example, a set-universe with relations defined on it, and among them the relation `size-of`, which takes a relation as its argument and returns the cardinal corresponding to the size of the relation. (Structures of this kind are called *hierarchical databases*.) Clearly, `size-of` is not a well-founded relation, for it is a member of its own domain. This problem can be got around within classical first-order theories, but at considerable cost in naturalness [3]. A coherent theory of hypersets would make modelling such structures a considerably easier task.

It is perhaps pertinent, before we begin, to say something about the Russell paradox, since there is a common misconception that it precludes a consistent theory of hypersets. That this *is* a misconception is easily demonstrated.

Suppose we are given a set x, well-founded or not. By the axiom of separation, we can form the Russell set $r_x = \{y \in x \,|\, y \notin y\}$. If x is well-founded, this just gives us x back again, i.e., $r_x = x$. But suppose x is a hyperset. Then the Russell antimony simply tells us that $r_x \notin x$, for otherwise we would have $r_x \in r_x$ if and only if $r_x \notin r_x$.

The following table gives r_x for several hypersets x:

x	$r_x = \{y \in x \mid y \notin y\}$
$x = \{x\}$	$r_x = \emptyset$
$x = \{1, x\}$	$r_x = \{1\}$
$x = \{0, \{1, x\}\}$	$r_x = \{0, \{1, x\}\} = x$

The Russell antimony has several consequences of importance, including that no set can contain all of its subsets as members, and that therefore the collection of well-founded sets, for instance, is not a set but a proper class.

None of these consequences, however, bears directly on the potential for a consistent theory of hypersets. It is to laying the groundwork for such a theory that we now turn.

3 Picturing Sets

How we distinguish distinct members among a class of like objects is a question that is fundamental in mathematics. In set theory it is a conspicuously practical question. Of course, when all sets in our universe are well-founded, we define equality among sets by the axiom of extensionality.

Definition 3.1 (Extensionality)
A binary relation R on a universe V is *extensional* if and only if

$$(\forall x)(\forall y)(x = y) \iff (\forall z)(zRx \iff zRy).$$

Thus, in Zermelo-Fraenkel set theory (ZF), we take the membership relation '$\in$' to be extensional, which in turn gives us a well-defined notion of set equality. (Sets are the same precisely when they have the same elements.) However, extensionality of this kind only works with well-founded relations. In the absence of an axiom of foundation, some new axiom, an *anti-foundation axiom*, will be required to enable us to define a rigorous notion of set equality that encompasses non-well-founded as well as well-founded sets. Consider for example the following hypersets (called *Quine atoms*):

$$a = \{a\} \qquad b = \{b\}$$

Are these the same set, or distinct sets? Definition 3.1 is no help here, as it gives us $a = b$ if and only if $a = b$. So the question remains: are there many Quine atoms, or only one? How we choose to answer this and related questions (and we are free to choose, as it happens) will determine the form our anti-foundation axiom is to take.

First we need some machinery. Although it is not strictly necessary to the development of anti-foundation axioms, for clarity and ease of presentation I will follow Aczel and others in the use of accessible pointed graphs (apg's) to lay the necessary groundwork.

Definition 3.2 (Accessible Pointed Graphs)
An *accessible pointed graph* G is a set of *nodes*, together with a set of *edges*, each of which consists of an ordered pair of nodes of G, such that,

i) there is a distinguished node, called the *point*, and

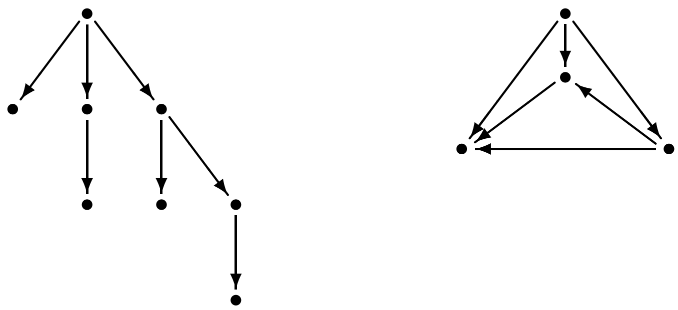


Figure 1: Two apg's.

ii) for every node n_0 there is a *path* of edges of the form

$$(n_k, n_{k-1})(n_{k-1}, n_{k-2}) \cdots (n_2, n_1)(n_1, n_0),$$

where n_k is the point.

We will denote the edge (n, n') by $n \to n'$, and say that n' is a *child* of n, and conversely that n is a *parent* of n'. For a given n, the set of children of n will be denoted ch_n, i.e., $n' \in \mathrm{ch}_n$ if $n \to n'$. If, for a node n, $\mathrm{ch}_n = \emptyset$, we say that n is *childless*. A path of edges from a node n_k to a node n_0 will be denoted by $n_k \to n_{k-1} \to \cdots \to n_0$. When there is such a path from a node n_k to a node n_0, we will say that n_k is an *ancestor* of n_0. The collection of all nodes of which n is an ancestor will be denoted $\overline{\mathrm{ch}_n}$.

Two apg's are represented in Figure 1.

There is nothing in the definition to prevent an apg having infinite paths, and this is one of the features that will make them useful in our development of a theory of hypersets. However, we want at this point to make the distinction precise, and to develop some more machinery for talking about apg's in general.

Definition 3.3 (Well-founded apg's)

A *well-founded apg* is an apg in which every subgraph has a least member under the "parent of" relation.

Definition 3.4 (Tagged apg)

Let G be an apg, and let $\mathcal{U}$ be a collection containing the empty set and (possibly) one or more *urelements*, where these latter are taken to be distinct, primitive objects having no structure. A *tagging* of G is a function $t : G \longrightarrow \mathcal{U}$ that assigns to each childless node of G exactly one element of $\mathcal{U}$. An apg G together with a tagging function t is said to be a *tagged apg.*

In what follows, we will assume whenever necessary the existence of a given collection $\mathcal{U}$ of tags. $\mathcal{U}$ may, of course, be a proper class.

Definition 3.5 (Decorated apg)
A *decorated apg* is a tagged apg G together with a recursively defined function d defined on each node n of G such that

$$d`n = \begin{cases} t`n & \text{if } n \text{ is childless,} \\ \{d`m \mid m \in \mathrm{ch}_n\} & \text{otherwise.} \end{cases}$$

Thus, a decoration of a node n is the set of decorations of all the nodes of which it is the parent, or simply its tag if it is childless.

Remark 3.6
It is easily seen that a well-founded, tagged apg is an extensional structure in the sense of Definition 3.1, where the extensional relation is taken to be "parent of."

Proposition 3.7 (Mostowski's Collapsing Lemma)
Every well-founded, tagged apg has a unique decoration.

Proof: Immediate from Definition 3.5, in which the decorating function d is defined by recursion on the well-founded relation "parent of." □

Definition 3.8 (Pictures)
Let G be a tagged apg with point n. If $d`n$ is a decoration of $n$, we say that $G$ *pictures* the set $d`n$.

Corollary 3.9
Every well-founded, tagged apg is a picture of a unique set.

We will use the notation G_a to mean that the apg G pictures the set a.

Proposition 3.10
Every well-founded set has a picture.

Proof: Given a set x, construct an apg G by assigning one node to each element in $TC`x$ (the transitive closure of $x$), and taking as edges all ordered pairs $(u, v)$ such that $u, v \in TC`x$ and $v \in u$. The set x is then the unique decoration of the point of G and hence, by Definition 3.8, G pictures x. □

Definition 3.11 (Canonical pictures and trees)
The picture constructed in the proof of Proposition 3.10 is called the *canonical picture* of x. The *unfolding* of the canonical picture G of x into the *canonical tree picture* G' of x is given by taking as its nodes the finite paths of G that start from the point of G, and whose edges are pairs of paths of G of the form

$$(x \to \cdots \to u,\ x \to \cdots \to u \to v).$$

As is evident from its construction, the canonical picture G of a set x is characterized by the property that no two nodes of G have the same decoration, i.e., the decorating function d is injective. Such a picture is also called an *exact picture*. The canonical tree picture G' of x is characterized by the property that no node of G' has more than one parent.

Example: Let $\mathcal{U} = \{\emptyset\}$, that is, every childless node will be tagged with the empty set. Then the two apg's in Figure 1 each picture the Von Neumann ordinal 3. The apg on the right is the canonical picture, and that on the left the canonical tree picture.

We now have that sets "are" pictures, and that pictures "are" sets. However, we still have that, in general, to every set there correspond *many* pictures. We now define the obvious equivalence relation on well-founded apg's.

Definition 3.12
Let $\mathcal{G}$ be the class of all well-founded, tagged apg's. Define the class-relation (equivalence class) $\sim_G$ on $\mathcal{G}$ by $G \sim_G G'$ if and only if G and G' picture the same set.

The class $\mathcal{G}/\sim_G$ is now a clear candidate for a model of ZF, as there is an obvious, one-to-one correspondence between its members and the sets x such that ZF $\vDash x$. However, before we present the result formally, it will be useful to extend our conception of apg's to encompass proper classes.

Definition 3.13 (Systems)
Let M be a collection (possibly but not necessarily a proper class) of nodes, together with a binary relation (respectively class-relation) R defined on it. We will say that M is a *system*, denoted $\langle M, R\rangle$, if for every node $n \in M$, the relation R interpreted as the "parent of" relation makes n the point of a tagged apg $G \subseteq M$, i.e., if the set of nodes $n \cup \{m \in M \mid m \in \overline{\mathrm{ch}_n}\}$ together with the edges corresponding to R is an apg. We will say that $\langle M, R\rangle$ is a *well-founded system* if every such $G \subseteq M$ is well-founded.

Thus, every apg is a system, but not every system is an apg, precisely analogously to the distinction between sets and classes. We are now in a position to view the class $\mathcal{G}$ defined above through this lens:

Definition 3.14 (Full systems)
Let $\langle M, R\rangle$ be a well-founded system. We will say that M is *full* if the map $\{G \subseteq M\} \longrightarrow \mathcal{G}/\sim_G$ given by $G \mapsto [G]$ is bijective.

Proposition 3.15 (Modelling Lemma)
Every well-founded, full system M is a model of ZFU (ZF with urelements). *A fortiori*, this will establish the corresponding result for ZF, i.e., when the tagging collection $\mathcal{U}$ contains only the empty set.

Proof:
The axioms of *pairing, sumset, power set, empty set, and infinity* are immediate from Proposition 3.10, the result that every well-founded set has a picture.

Extensionality: By induction on the well-founded relation "parent of." Suppose $n, m \in M$ such that $\mathrm{ch}_n = \mathrm{ch}_m$. Then $d`n = d`m$, hence $G_{d`n} \sim_G G_{d`m}$. Since M is full, this means $G_{d`n} = G_{d`m}$, so $n = m$.

Foundation is built into the definition of M.

Separation: Let $n \in M$, and let $\phi(x)$ be a formula, possibly containing elements of M, with at most x free. Let $a = \{d`m \mid m \in \mathrm{ch}_n \wedge \phi(m)\}$. Now a is clearly a set, and hence has a unique picture $G \subseteq M$.

Replacement: Suppose the antecedent, i.e., let $\phi(x, y)$ be any formula such that to each $n \in M$ there corresponds a unique $m \in M$ such that $\phi(n, m)$. Now let $n \in M$, and set $a = \{d`m \mid \phi(n', m) \text{ for some } n' \in \mathrm{ch}_n\}$. As before, a is clearly a set, and hence has a unique picture in M. $\square$

Remark 3.16

A full system M also supports the axiom of choice: Let $\{n_i\}$ be a collection of nodes of M such that ch_n is non-empty for each i. Then for each i, $d`n_i$ is a non-empty collection of sets. By the axiom of choice, we can form a set $\{d`m_i \mid d`m_i \in d`n_i$ for each $i\}$. As always, this set has a picture in M. Thus, M is also a model of ZFC(U), that is, ZF(U) with the axiom of choice.

4 Extensionality on Hypersets

Although routine and even somewhat prosaic, the development in Section 3 of accessible pointed graphs to build a model of Zermelo Fraenkel set theory will serve a very useful purpose in the development of our antifoundation axioms. This is because we can use apg's to picture non-wellfounded as well as well-founded sets. The fact that when we restrict our model to well-founded apg's we get the classical theory back again indicates that hyperset theory is an extension of, not a replacement for, classical set theory, and this will be formally demonstrated at the end of this section, together with a generalized modelling lemma for hyperuniverses. On one hand, this means that hypersets will tell us nothing new about well-founded sets. On the other hand, it will make it a straightforward matter to demonstrate the equiconsistency of hyperset theory with ZF.

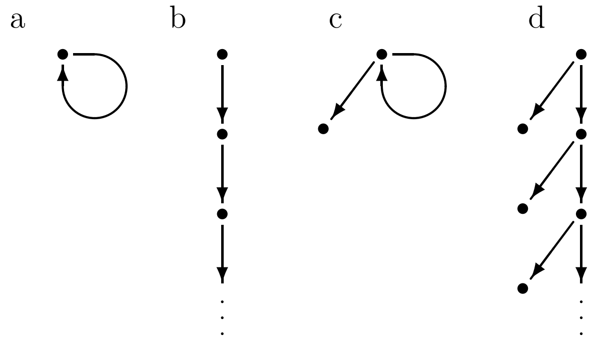


Figure 2: Non-well-founded apg's

We saw in Section 2, using the example of Quine atoms, that the fundamental issue to be confronted in any theory of hypersets is that of extensionality. We now turn our full attention to this issue. Consider the apg's represented in Figure 2. The apg (a) is a (canonical) picture of a Quine atom, i.e., a set of the form $x = \{x\}$, and the apg (b) is its "unfolding" into the canonical tree picture. The apg (c) is a picture of a set of the form $x = \{0, x\}$, and (d) is its tree picture. Since these are non-well-founded apg's, we no longer have at our disposal results like Proposition 3.7, which guarantees for well-founded apg's that the decoration of each graph will be unique. Thus, the question remains to be answered whether, for instance, the apg's in Figure 2 each represent none, one, or many sets.

Let us begin by enlarging our old collection $\mathcal{G}$ of well-founded apg's to a collection $\mathcal{H}$ that includes all non-well-founded apg's. Note that the definition of a system M remains the same, but that our systems are no longer necessarily well-founded. Also, in what follows, we will use both "Gn" and "Mn" to denote a graph $G \subseteq M$ with point n. (This should be distinguished from the notation "G_a," which we will continue to use to denote any graph that is a picture of the set a.)

With our enlarged collection $\mathcal{H}$ we naturally lose all of the results of Section 3 that depend on foundation. However, we retain one very crucial result:

Proposition 4.1
Every set has a picture.

Proof: Exactly the same as Proposition 3.10, except that the decoration is no longer necessary unique.

We now go to work to make up for the loss of the other results of Section 3. We begin by defining a class of functions on systems.

Definition 4.2 (System map)
Let M and M' be systems. A map $\pi : M \longrightarrow M'$ is a *system map* if for all n in M,

$$\mathrm{ch}_{\pi n} = \{\pi m \,|\, m \in \mathrm{ch}_n\}.$$

If π is a bijection then it is a *system isomorphism*, and when such exists we write $M \cong M'$.

The special kind of relation known as a *bisimulation* will be central to our development of suitable notions of extensionality on hypersets.

Definition 4.3 (Bisimulation)
A binary relation R on M is a *bisimulation* if $R \subseteq R^+$, where for $n, m \in M$, nR^+m if and only if

$$(\forall x \in \mathrm{ch}_n)(\exists y \in \mathrm{ch}_m)(xRy) \wedge (\forall y \in \mathrm{ch}_m)(\exists x \in \mathrm{ch}_n)(xRy).$$

Observe that $R_0 \subseteq R \Rightarrow R_0^+ \subseteq R^+$, i.e., the operator $(\ )^+$ is monotone.

We can now use bisimulations to define a class of equivalence relations on $\mathcal{H}$.

Definition 4.4 (Regular bisimulation)
A bisimulation relation $\sim$ on $\mathcal{H}$ is a *regular bisimulation* relation if

i) $\sim$ is an equivalence relation on $\mathcal{H}$,

ii) $Gn \cong G'n' \Rightarrow Gn \sim G'n'$, and

iii) $\mathrm{ch}_n = \mathrm{ch}_{n'} \Rightarrow Gn \sim Gn'$ for $n, n' \in G$.

Definition 4.5 ($\sim$ – extensional)
Let $\sim$ be a regular bisimulation. Then a system M is a $\sim$ – *extensional* system if for all $n, m \in M$,

$$Mn \sim Mm \Rightarrow n = m.$$

Proposition 4.6
If M is $\sim$ – extensional, then for any system M_0 there is at most one injective system map $\pi : M_0 \hookrightarrow M$.

Proof: Suppose $\pi_1, \pi_2 : M_0 \hookrightarrow M$, and let $n_0 \in M_0$. Since π_1 and π_2 are injective, we have $\pi_2\pi_1^{-1} : M(\pi_1 n_0) \longrightarrow M(\pi_2 n_0)$ is a system isomorphism, hence $M(\pi_1 n_0) \cong M(\pi_2 n_0)$. By Definition 4.4 this gives $M(\pi_1 n_0) \sim M(\pi_2 n_0)$, so $\pi_1 n_0 = \pi_2 n_0$. The choice of n_0 was arbitrary, so $\pi_1 = \pi_2$. □

Definition 4.7 ($\sim$ – complete)
A system M is $\sim$ – *complete* if it is $\sim$ – extensional and every $\sim$ – extensional graph $G \in \mathcal{H}$ is system isomorphic to Mn for some n in M. (By the above proposition, such an isomorphism is necessarily unique. An immediate consequence is that for n, m in a $\sim$ – complete system M, $\mathrm{ch}_n = \mathrm{ch}_m \Rightarrow n = m$.)

Proposition 4.8
Every $\sim$ – complete system M is full.

Proof: Recall that M is full when the map given by $G \mapsto [G]$ is a bijection, where $G \subseteq M$ and $[G] \in \mathcal{H}/\sim$. To show that this map is surjective, it is sufficient, in light of Definition 4.7, to show that every equivalence class $[G]$ contains a $\sim$ – extensional member. So suppose G is a graph. We can form the $\sim -$extensional graph G' by identifiying all nodes n, m in G such that $Gn \sim Gm$. Evidently $G \sim G'$, so $G' \in [G]$. For "into," suppose Gn, Gm map to $[G]$. Then $Gn \sim Gm$, whence $n = m$ since Gn and Gm are $\sim -$extensional. □

What these results show is that we can use apg's to build a model of any set universe $V^{\sim}$ corresponding to $\mathcal{H}/{\sim}$, where $\sim$ is a regular bisimulation on the universe $\mathcal{H}$ of all apg's. Extensionality on the hyperset universe $V^{\sim}$ is defined in the obvious way: If x, y are sets, then $x = y$ whenever $G_x \sim G_y$ for canonical pictures G_x, G_y of the sets x, y respectively.

Definition 4.9 (Anti-foundation axiom schema)
Let $\sim$ be a regular bisimulation on the collection $\mathcal{H}$ of apg's. We will denote by AFA$^{\sim}$ the axiom corresponding to the statement that an apg is an exact picture if and only if it is $\sim -$extensional. (Recall that an apg is an exact picture if and only if each node has a unique decoration.)

To complete this development, we turn now to the generalized modelling lemma for hyperuniverses. We will split this into two parts, dealing first with a general lemma for ZF$^-$, that is, ZF without an axiom of foundation.

Theorem 4.10 (Rieger's Theorem)
Every full system is a model of ZF$^-$.

Proof: Let M be a full system. Then for any subset x of M there is a unique $n \in M$ such that $x = \mathrm{ch}_n$. We denote this n by x^M. With this notation in hand, we turn to the axioms:

Extensionality: Let $n, m \in M$ such that

$$M \vDash (\forall x)(x \in n \iff x \in m).$$

Then $\mathrm{ch}_n = \mathrm{ch}_m$. Since $n = (\mathrm{ch}_n)^M$ and $m = (\mathrm{ch}_m)^M$, we have $M \vDash n = m$.

Pairing, Union, Power Set, and Infinity follow precisely as in Proposition 3.15.

Separation: Let $n \in M$, and let $\phi(x)$ be a formula, possibly containing constants for elements of M, with at most the variable x free, and set

$$n' = \{m \in \mathrm{ch}_n \mid M \vDash \phi(m)\}^M.$$

Then

$$M \vDash \forall x(x \in n' \iff x \in n \wedge \phi(x)).$$

Collection: Let $n \in M$, and let $\phi(x, y)$ be a formula, possibly containing constants for elements of M, with at most the variables x and y free, and suppose that

$$M \vDash (\forall x \in Mn)(\exists y)\phi(x, y).$$

Then

$$(\forall x \in \mathrm{ch}_n)(\exists y)(y \in M \wedge M \vDash \phi(x, y)).$$

By the Collection Schema, there is a set m such that

$$(\forall x \in \mathrm{ch}_n)(\exists y \in m)(y \in M \wedge M \vDash \phi(x, y)).$$

Let $n' = (m \cap M)^M$. Then

$$M \vDash (\forall x \in Mn)(\exists y \in Mn')\phi(x, y).$$

□

(We observe here that the Axiom of Choice may easily be obtained if desired, and this is left to the reader, or see [6].)

To get the rest of the generalized modelling result, we need to develop a little more machinery. In particular, we must assume that we are given a definition in the language of set theory of a regular bisimulation $\sim$, call it $\phi(x, y)$, without any parameters and having at most the variables x, y free. Using this definition we may form a sentence that defines when a system M is $\sim -$ complete, and we will call that sentence, as above, AFA$^{\sim}$.

Now let M be a full system, and let $\sim_M$ be the relation on M that the definition $\phi(x, y)$ of $\sim$ defines in M. We can represent an apg c in M as a triple $((a, b), u)$, where a is a set, b is a binary relation on a, and u is an element of a. Then, if $c \in M$, we have $M \vDash$ "c is a pointed graph" if and only if $c = ((a, b)^M, u)^M$ for some (uniquely determined) $a,\ b,\ u \in M$ such that

$$\mathrm{ch}_b \subseteq \{(x, y)^M \mid x, y \in \mathrm{ch}_a\}$$

and $u \in \mathrm{ch}_a$. With such a c in M we may now associate the pointed graph

$$((\mathrm{ch}_a, \{(x, y) \mid (x, y)^M \in \mathrm{ch}_b\}), u).$$

Call this pointed graph ext_c.

Definition 4.11 (Absolute formula)

We will say that the regular bisimulation $\sim$ is *absolute* for a system M if for all $c, d \in M$ such that $M \models$ "c, d are apg's," we have

$$c \sim_M d \iff ext_c \sim ext_d.$$

Theorem 4.12 (Modelling lemma for AFA$^{\sim}$)

Let $\sim$ be a regular bisimulation whose definition is absolute for full systems. Then each $\sim -$complete system M is a full model of ZF^{-}+AFA$^{\sim}$.

In the next section we develop three distinct regular bisimulations, and it is a notable fact that the restriction of these (or indeed of any) regular bisimulations to well-founded apg's is equivalent to the relation $\sim_G$ of Section 3. This is a ready consequence of well-foundedness. (For clarity, we restrict our attention to the case where all childless nodes are tagged with the empty set. The case where urelements are included is similar.)

Lemma 4.13 (Uniqueness of $\sim_G$)

The relation $\sim_G$ of Definition 3.12 on the class $\mathcal{G}$ of all well-founded apg's is the unique regular bisimulation on $\mathcal{G}$.

Proof: Let M be a well-founded system, $n, n' \in M$. By definition of $\sim_G$, $n \sim_G n' \iff d\text{‘}n = d\text{‘}n'$. By definition of the decorating function d, we then have

$$n \sim_G n' \iff \{dm \mid n \to m\} = \{dm' \mid n' \to m'\}.$$

In other words, $\sim_G$ satifies

$$n \sim_G n' \iff (\forall x \in \text{ch}_n)(\exists y \in \text{ch}_{n'})(x \sim_G y) \wedge (\forall y \in \text{ch}_{n'})(\exists x \in \text{ch}_n)(x \sim_G y).$$

It is also an obvious equivalence relation, and the other conditions of Definition 4.4 are likewise immediate. Thus, $\sim_G$ is a regular bisimulation.

For uniqueness, we have that given any well-founded system M, the childless nodes are all isomorphic trivially, so for all childless nodes n, n' we have $n \sim n'$ for any regular bisimulation $\sim$. By induction, the regular bisimulation is now fully determined, and must therefore be $\sim_G$. □

Thus, as before noted, each $V^{\sim}$ will be an expanded universe, in the sense that it will properly contain the ZF universe V.

In general, there are many regular bisimulations on $\mathcal{H}$, and it is in defining them that we discover our anti-foundation axioms. We turn now to a (concise) exposition of those that are in the literature.

5 Anti-foundation axioms

We here summarize three distinct anti-foundation axioms. The first is due to Forti and Honsell [11], who called it X_1. It is better known in the literature as AFA, a terminology due to Aczel. The second is based upon an axiom scheme of P. Finsler [7], and is denoted FAFA. The third, SAFA, is due to Dana Scott [17].

We give a brief presentation of each axiom, and then demonstrate some of the crucial differences among them, in particular showing that they are pairwise inconsistent.

AFA

We say that a relation R on a system M is *small* if R is a set. Define $\sim_A$ on a system M by $Mn \sim_A Mm$ if nRm for some small bisimulation R.

Proposition 5.1
Let M be any system. Then $\sim_A$ is the unique maximal bisimulation on M. Moreover, $\sim_A$ is a regular bisimulation.

Proof: Let n, m in M such that $Mn \sim_A Mm$. Then nRm for some small bisimulation R. By definition of $\sim_A$ and the fact that the operation $(\ )^+$ is monotone, we have

$$(nRm \Rightarrow Mn \sim_A Mm) \Rightarrow (nR^+m \Rightarrow Mn \sim_A^+ Mm).$$

But R is a bisimulation, so $R \subseteq R^+$. In particular, then, nR^+m and so $n \sim_A^+ m$, which implies $\sim_A \subseteq \sim_A^+$. Therefore, $\sim_A$ is a bisimulation.

For uniqueness, let R be any bisimulation on M such that nRm. Then $R' = R \cap (Mn \times Mm)$ is a small bisimulation such that $nR'm$, so $n \sim_A m$.

$\sim_A$ is a regular bisimulation: The identity map on any system is an obvious bisimulation; if R is a bisimulation then R^{-1} is a bisimulation; the composition of two bisimulations is a bisimulation. Thus, $\sim_A$ is reflexive, symmetric, and transitive, i.e., an equivalence relation. Any system isomorphism π defines a bisimulation R on M, where xRy if and only if $x \in Mn$, $y \in Mm$, and $\pi x = y$, so $Mn \cong Mm \Rightarrow n \sim_A m$. Finally, suppose $\mathrm{ch}_n = \mathrm{ch}_m$. Define R on M by

$$R = \{(n, m)\} \cup \{(x, x) \mid x \in Mn\}.$$

Then nRm, hence $n \sim_A m$. □

What kind of anti-foundation axiom does $\sim_A$ imply? Let us define extensionality on sets by $a = b$ if and only if $G_a \sim_A G_b$ for canonical pictures G_a, G_b. Then the maximality of $\sim_A$ tells us that sets will be equal whenever possible. More specifically, it tells us that sets are the same if and only if they can be pictured by the same apg. This leads to a formulation of the following anti-foundation axiom:

Definition 5.2 (AFA)
An apg is an exact picture if and only if it is $\sim_A$−extensional.

Corollary 5.3
AFA $\Rightarrow$ every apg has a unique decoration, i.e., pictures a unique set.

AFA has consequences that will be examined in more detail in the final section. One consequence that may immediately be noted is that the Quine atom is unique, which we will follow Aczel and others in denoting Ω. That is, Ω is the unique set such that $\Omega = \{\Omega\}$.

<u>FAFA</u>

For Finsler's anti-foundation axiom, we begin by defining a construction on a system M in the following way. Suppose Mn is a graph in M. Let $(Mn)^*$ be the graph consisting of the nodes and edges of Mn that are on paths starting from some child of n, together with a new node $*$ and a new edge $(*, m)$ for every $m \in \mathrm{ch}_n$. Note that if n does not lie on any path starting from a child of n then $(Mn)^*$ will be isomorphic to Mn via an isomorphism that is the identity on all $m \in \overline{\mathrm{ch}_n}$ and which maps $*$ to n. If n does lie on such a path then $(Mn)^*$ consists of the nodes and edges of Mn together with the new node and edges.

We now define $\sim_F$ on a system M by $n \sim_F m$ if and only if $(Mn)^* \cong (Mm)^*$. Note that if $\mathrm{ch}_n = \mathrm{ch}_m$ then $(Mn)^* = (Mm)^*$ and hence $(Mn)^* \cong (Mm)^*$ trivially.

Proposition 5.4
The relation $\sim_F$ is a regular bisimulation.

Proof: Suppose $Mn \sim_F Mm$. Then $(Mn)^* \cong (Mm)^*$. Obviously then, for every $n' \in \mathrm{ch}_n$, there is an $m' \in \mathrm{ch}_m$ such that $Mn' \cong Mm'$, and hence $(Mn')^* \cong (Mm')^*$. By symmetry, this establishes that $\sim_F$ is a bisimulation.

That $\sim_F$ is an equivalence relation is immediate from the fact that it is defined on an isomorphism. For instance, for symmetry,

$$\begin{aligned} Mn \sim_F Mm &\iff (Mn)^* \cong (Mm)^* \\ &\iff (Mm)^* \cong (Mn)^* \\ &\iff Mm \sim_F Mn \end{aligned}$$

Transitivity follows similarly.

We saw already that $\mathrm{ch}_n = \mathrm{ch}_m \Rightarrow Mn \sim_F Mm$.

Finally, suppose $\pi : Mn \cong Mm$. Then we can define $\pi' : (Mn)^* \cong (Mn)^*$ by $\pi'(*) = *$, and $\pi' = \pi$ otherwise. Thus, $\sim_F$ is a regular bisimulation. □

This leads us to a formulation of a new anti-foundation axiom:

Definition 5.5 (FAFA)
An apg is an exact picture if and only if it is $\sim_F$–extensional.

SAFA

Recall that a *tree* is an apg with the property that no node has more than one parent. Any graph Mn in a system M can be turned into a tree by "unfolding" it in the manner of Definition 3.11. We will denote such an unfolding of a graph Mn by $(Mn)^t$. For Scott's anti-foundation axiom, we need the following notion.

Definition 5.6 (Irredundant tree)
A tree is said to be *irredundant* if it has no non-trivial automorphism.

Another way of putting this is to say that a graph Mn is redundant if there is a node $m \in Mn$ and nodes $x, y \in \mathrm{ch}_m$ such that $(Mx)^t \cong (My)^t$.

We now define $\sim_S$ on a system M by

$$Mn \sim_S Mm \iff (Mn)^t \cong (Mm)^t.$$

Proposition 5.7
Let M be a system. The relation $\sim_S$ on M is a regular bisimulation.

Proof: As with $\sim_F$, the proof follows readily from the fact that $\sim_S$ is defined by means of a system isomorphism. The proof is essentially identical to that of Proposition 5.4, except where it is more trivial. □

This leads us to the following anti-foundation axiom:

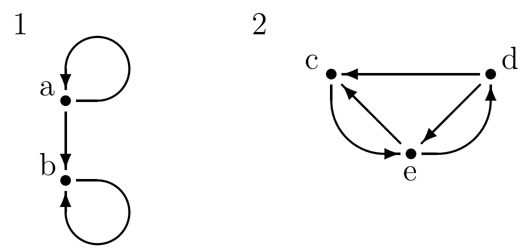
Definition 5.8 (SAFA)
An apg is an exact picture if and only if it is $\sim_S$ – extensional.

Corollary 5.9
A tree is isomorphic to a canonical tree picture if and only if it is irredundant.

We complete this presentation with a demonstration that these axioms are in fact distinct from one another.

Theorem 5.10
The antifoundation axioms AFA, FAFA, and SAFA are pairwise inconsistent.

Proof: Consider these apgs:



We begin by observing that neither of the above graphs is $\sim_A$ – extensional. In apg (1), Mb is a canonical picture of the Quine atom Ω, and thus we have

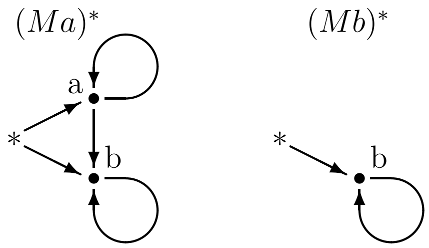
$$a = \{b, \{b\}\} = \{b, b\} = \{b\} = b = \Omega.$$

Similarly, for apg (2), we have

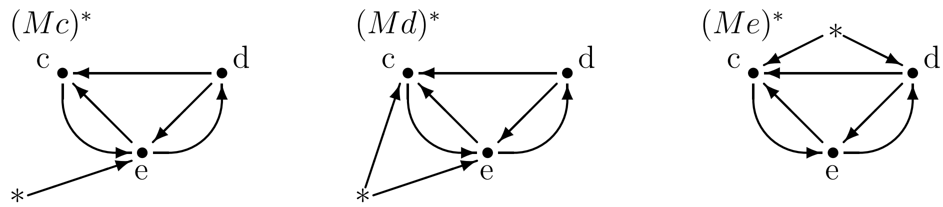
$$\begin{aligned} c &= \{e\} \\ &= \{\{c, d\}\} \\ &= \{\{c, \{c, e\}\}\} \\ &= \{\{c, \{c, \{c, \cdots\}\}\}\} \end{aligned}$$

Since $\Omega = \{\Omega\} = \{\Omega, \{\Omega, \{\Omega, \cdots\}\}\}$, we have that $c = \Omega$ is a decoration for c. By AFA, such a decoration is unique, so $c = d = e = \Omega$. (In fact, it is a straightforward matter to prove that, assuming AFA, an apg pictures Ω if and only if every node has a child [1].)

We next observe that both (1) and (2) are $\sim_F$−extensional. This is readily seen by forming for each apg the constructions described in the development of FAFA. For apg (1) the constructions are as follows.

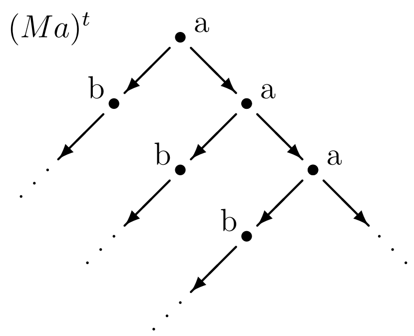


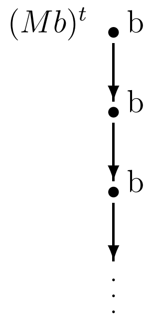
It is seen at once that $(Ma)^* \ncong (Mb)^*$. Similarly, for apg (2) we have,



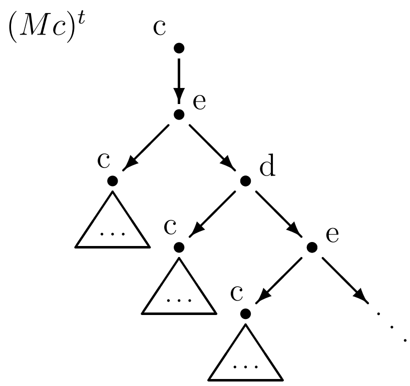
Again, it is clear that $(Mc)^*$, $(Md)^*$, and $(Me)^*$ are pairwise non-isomorphic.

Finally, we consider these apg's in the light of SAFA. The apg (1) is clearly $\sim_S$−extensional, for consider the unfoldings of Ma and Mb:





Obviously, $(Ma)^t \ncong (Mb)^t$. However, the apg (2) is *not* $\sim_S$−extensional, as may be seen from its unfolding:



Here, we see that $(Md)^t \cong (Me)^t$, so (2) is not an extensional apg under SAFA.

Summarizing, for these sample apg's we have,

	(1)	(2)
$\sim_A$ – extensional	NO	NO
$\sim_F$ – extensional	YES	YES
$\sim_S$ – extensional	YES	NO

Therefore, AFA, FAFA, and SAFA are pairwise inconsistent axioms. □

We saw before that the regular bisimulation $\sim_A$ corresponding to AFA was maximal. As the proof of the above theorem suggests, it turns out that the regular bisimulation $\sim_F$ is minimal on the universe of apg's, and $\sim_S$ is, so to speak, properly between them.

In the final section, we turn our attention to some applications of the hyperset theory ZFC$^-$+AFA.

6 Applying Hypersets

In this section we work solely in the theory ZFCU$^-$+AFA, that is, Zermelo Fraenkel set theory with choice and urelements, and the anti-foundation axiom AFA of the previous section. It is this theory which has the widest range of applications, because it ensures solutions to "systems of equations" of sets. This result has come to be called the *Solution Lemma*, a term coined by Barwise and Etchemendy [2].

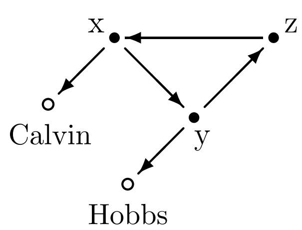
We begin with some examples. Consider for instance the following system of set equations,

$$\begin{aligned} x &= \{\text{Calvin}, y\} \\ y &= \{\text{Hobbs}, z\} \\ z &= \emptyset, \end{aligned}$$

where "Calvin" and "Hobbs" are taken to be urelements. This system can be solved by simple back substitution, putting $\emptyset$ in for z in the definition of y, yielding $y = \{\text{Hobbs}, \emptyset\}$, and substituting this into the definition of x to get $x = \{\text{Calvin}, \{\text{Hobbs}, \emptyset\}\}$. However, it is not obvious what to do with the next system:

$$\begin{aligned} x &= \{\text{Calvin}, y\} \\ y &= \{\text{Hobbs}, z\} \\ z &= \{x\} \end{aligned}$$

It is, however, simple to draw an apg picturing this set.



The axiom AFA tells us that this graph has a unique decoration, and evidently that decoration, if we can find it, will be the solution of our system.

In fact, "unfolding" x, we get

$$\begin{aligned} x &= \{\text{Calvin}, y\} \\ &= \{\text{Calvin}, \{\text{Hobbs}, z\}\} \\ &= \{\text{Calvin}, \{\text{Hobbs}, \{x\}\}\} \\ &= \{\text{Calvin}, \{\text{Hobbs}, \{\{\text{Calvin}, y\}\}\}\} \\ &= \{\text{Calvin}, \{\text{Hobbs}, \{\{\text{Calvin}, \{\text{Hobbs}, \cdots\}\}\}\}\} \end{aligned}$$

Consequently, by AFA, this is the unique decoration of the node labelled x, and we will see below that it therefore represents a solution for x in the system of equations. If we do the same for y and z, we will find that these nodes get distinct decorations, so evidently the above is an exact apg.

In a system of equations such as the above, we will refer to the $x, y, \ldots$ as *set indeterminates*. What the Solution Lemma says is that for any system of equations in indeterminates $x, y, z, \ldots$, say

$$\begin{aligned} x &= a(x, y, \ldots) \\ y &= b(x, y, \ldots) \\ z &= c(x, y, \ldots) \\ &\vdots \end{aligned}$$

where each indeterminate appears exactly once in the left-hand side of an equation, then the system has a unique solution in the universe of hypersets.

Given a collection $\mathcal{U}$ of urelements, we will write $V_{\mathcal{U}}$ for the hyperuniverse of sets with urelements from $\mathcal{U}$. Formally, we regard a collection of set indeterminates $\mathcal{X}$ as "extra urelements," and write $V_{\mathcal{U}}[\mathcal{X}]$ for $V_{\mathcal{U} \cup \mathcal{X}}$. (The notation is taken from ring theory; the construction is analogous to that of a ring of polynomials $\mathbb{F}[X]$ with indeterminates in X over a field $\mathbb{F}$.) This scheme is illustrated below.

By an equation in $\mathcal{X}$, we mean an expression of the form

$$x = a,$$

where $x \in \mathcal{X},\ a \in V_{\mathcal{U}}[\mathcal{X}]$. By a *system of equations* in $\mathcal{X}$ we mean a family of equations $\{x = a_x \mid x \in \mathcal{X}\}$, exactly one equation for each indeterminate $x \in \mathcal{X}$. By an *assignment* for $\mathcal{X}$ in $V_{\mathcal{U}}$ we mean a function $f : \mathcal{X} \longrightarrow V_{\mathcal{U}}$ which assigns an element $f(x)$ of $V_{\mathcal{U}}$ to each indeterminate $x \in \mathcal{X}$.

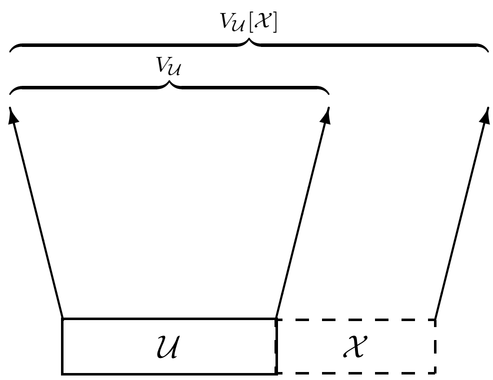


Figure 3: The Hyperuniverse.

As we saw above, any such assignment extends in a natural way to a function $\hat{f} : V_{\mathcal{U}}[\mathcal{X}] \longrightarrow V_{\mathcal{U}}$. Thus, given some $a \in V_{\mathcal{U}}[\mathcal{X}]$, one works with a canonical graph depicting a, replacing any childless nodes tagged by an indeterminate $x \in \mathcal{X}$ with a graph depicting the set $f(x)$. Typically, we write $a(x, y, \ldots)$ for $\hat{f}(a)$.

Thus, an assignment f is a *solution* of an equation $x = a(x, y, \ldots)$ if

$$f(x) = a(f(x), f(y), \ldots).$$

Theorem 6.1 (The Solution Lemma)

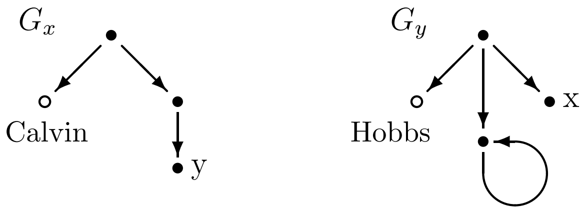
Every system of equations in a collection $\mathcal{X}$ of indeterminates over $V_{\mathcal{U}}$ has a unique solution.

Despite its intuitive appeal, a proper proof of the Solution Lemma requires a technical preparation which is too lengthy to permit its inclusion in this paper. The reader is referred, with apologies, to Aczel's book [1], or see also Devlin [6].

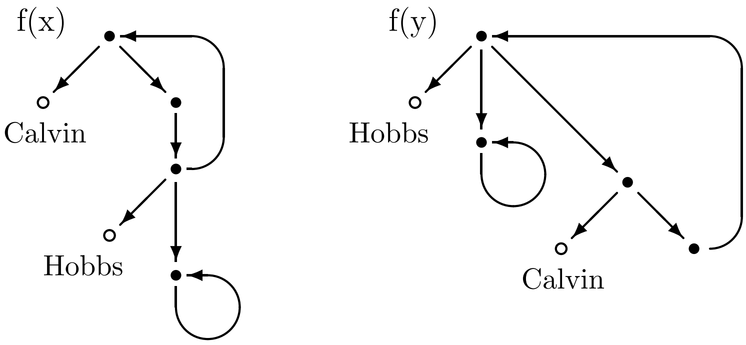
Example: Consider the following system of equations.

$$\begin{aligned} x &= \{\text{Calvin}, \{y\}\} \\ y &= \{\text{Hobbs}, \Omega, x\} \end{aligned}$$

The sets on the right-hand side of the equations are depicted below.



To depict the solutions to the equations, we replace any edges terminating in x with an edge terminating in the top node of G_x, and any edges terminating in y with an edge terminating in the top node of G_y. This gives us the following graphs:



By AFA, these graphs have unique decorations, and the sets assigned to the top nodes are solutions of our equations. Since any solution of the system of equations would give rise to decorations of these graphs, then AFA guarantees that the solution is unique.

Proposition 6.2
Assume the axioms of $ZFCU^-$. Then the Solution Lemma is equivalent to AFA.

Proof: In light of Theorem 6.1, we need only prove the forward direction (SL $\Rightarrow$ AFA). However, every tagged apg is a picture of some system of set equations, where each untagged node corresponds to a set indeterminate. Since by the Solution Lemma the system pictured has a unique solution, then the apg has a unique decoration, i.e., pictures a unique set. $\square$

The Solution Lemma is central to the application of the hyperset theory (with AFA) in many fields. Barwise and Etchemendy have used it to analyse Russellian propositions [2], Aczel demonstrates its use in computer science (specifically, the Synchronous Calculus of Communicating Systems) [1], Tanaka and Tsujishita use it to determine the orbits of a knowledge state space (modelled as a dynamical system) to find a solution for the muddy boys puzzle [18], while others are examining its use as a tool in other fields from classical geometry to modern philosophy [13].

In this paper I have sought to present the elements of the theory in such a way as to highlight both its rigor and its naturalness. In doing so, I have avoided many interesting side issues, and left unsaid many interesting points, such as that the universe $V^{\sim_A}$ is "final" in the category of universes of set theory, for instance [1]. There is also much to be said, or perhaps answered, regarding its relation to other unconventional set theories, such as set theory with a universal set.

Although the primary sources are excellent, I believe there is a need for a comprehensive general text on the subject, perhaps as part of a broader treatment of extensions of classical set theory.

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